Noise in Amplified $\mathscr{PT}$–Symmetric Systems

by

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Abstract

This thesis is primarily a study of noisy circuits focusing on noise in a $\mathscr{PT}$–symmetric circuit introduced by the presence of the gain and loss elements. In chapter 1, we give an introduction to thermal noise and $\mathscr{PT}$–symmetric electronics, providing all the calculations for the derivation of noisy circuits' signal-to-noise ratio (SNR). In chapter 2, we focused on the main calculation of noisy $\mathscr{PT}$–symmetric RLC circuits in comparison with noisy non–$\mathscr{PT}$–symmetric RLC circuits. Chapter 3 is devoted to the new insight of having a less noisy circuit by using a parametrically pumped LC circuit to manipulate noise.

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Chapter 1

Introduction

The presence of the noise in electronic devices is unavoidable so that measuring the effects of noise sources is imperative in studying circuits. There are not enough study on the effects of thermal noise in $\mathscr{PT}$–Symmetric electronic circuits. because the whole point of having such non–Hermitian systems is for their applications in enhancing the sensitivity of the electronic [9, 13], mechanical [17, 18] and optical [14, 15] devices. Moreover, with the advent of topology in physics, in particular, condensed matter physics, topological phases have been studied extensively in non–Hermitian systems [19–24]. In order to study the effect of noise in $\mathscr{PT}$–Symmetric electronic circuits we need first to understand noisy non-$\mathscr{PT}$–Symmetric systems to make a good comparison between these two circuits. This comparison is crucial since we have to understand if having a noisy $\mathscr{PT}$–Symmetric system can still remain as a good option for enhancing the sensitivity of devices. In this chapter the concepts of thermal noise and $\mathscr{PT}$ symmetry in electronics are introduced.

1.1 Noise Power Spectral

Noise is a (or a stochastic) random process. For a classical random signal time-dependent $V(t)$, its mean value over sampling time T is zero $\langle V(t)\rangle =$

$\frac{1}{T}\int_T V(t)dt = 0$ due to the randomness. However, the RMS of the signal $\langle V(t)^2\rangle$ is not zero, $\langle V(t)^2\rangle = \frac{1}{T}\int V(t)^2 dt > 0$. This characteristic is resemblance to Gaussian function, that is why it is called it white Gaussian noise. For the noise analysis calculating power spectral density (PSD) is crucial and it depends on autocorrelation function. We can define an autocorrelation function as follows

$$G_V = \langle V(t)V(t')\rangle \tag{1.1}$$

Autocorrelation function would tell us about how the fluctuations of $V(t)$ at time t correlate with $V(t')$ at time $t' = t - \tau$ where $t \neq t'$, and τ is just the difference between two different times $\tau = t - t'$. Autocorrelation function is stationary if it is time translation invariant meaning that it just depends on the difference of times, $G_V(t,t') = G_V(t-t')$. Moreover, there exists τ_c correlation such that $|t - t'| \gg \tau_C$ then we have $G_V(t,t') \to 0$ as a result the signals at different times become uncorrelated. For noise analysis we need to measure the spectral density of the noise, since it is easier to measure in frequency domain we have to Fourier transform $V(t)$ as follows [1]:

$$\begin{aligned} V_T(\omega) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dtV(t)e^{-i\omega t} \\ V(t) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} d\omega V_T(\omega)e^{i\omega t} \end{aligned} \tag{1.2}$$

where T is the sampling time. So the correlation function can be defined as follows:

$$G_V(\tau) = \lim_{T\to\infty}\frac{1}{T}\int_t^{t+\tau} dtV(t)V(t+\tau) \tag{1.3}$$

Power Spectral Density (PSD) of the noise measures noise power in a frequency range and is defined [26]

$$S_V(\omega) = \lim_{T\to\infty} \frac{2|V_T(\omega)|^2}{T} \tag{1.4}$$

If we take a Fourier transform of eq. (1.3) then we have:

$$\mathscr{G}_V(\omega) = \lim_{T\to\infty} \frac{|V_T(\omega)|^2}{T} \tag{1.5}$$

So that $S_V(\omega) = 2\mathscr{G}_V(\omega)$. As an example, since a white Gaussian noise has a zero mean value or zero autocorrelation function so the autocorrelation function can be defined as $G_V = \sigma^2\delta(t)$ where σ^2 is the variance in Gaussian function, and $\delta(t)$ is the delta function, and as a result PSD for white noise is $S_v = \sigma^2$. It means that the spectral density is flat for white noise and it does not depend on frequency. Based on Parseval's theorem between the signal and its Fourier transform $\int_{-\infty}^{\infty} dt|V(t)|^2 = \int_{-\infty}^{\infty} d\omega|V_T(\omega)|^2$, for calculating the averaged power P of the signal which is $P = \lim_{T\to\infty} \frac{1}{T}\int_t^{t+\tau} dt|V(t)|^2$, we have:

$$P = \int_{-\infty}^{\infty} dt|V_T(\omega)|^2 \tag{1.6}$$

The noise power, $\langle V_n^2\rangle$, can be calculated in terms of the power spectral density as

$$\langle V_n^2\rangle = \int_0^{\infty} df S_V(f) = \int_0^{\infty} \frac{d\omega}{2\pi} S_V(\omega) = \int_{-\infty}^{+\infty} d\omega|V_T(\omega)|^2 \tag{1.7}$$

Where S_V is the power spectral density, in the units of Watts per Hz.

1.2 Johnson-Nyquist Noise or Thermal Noise

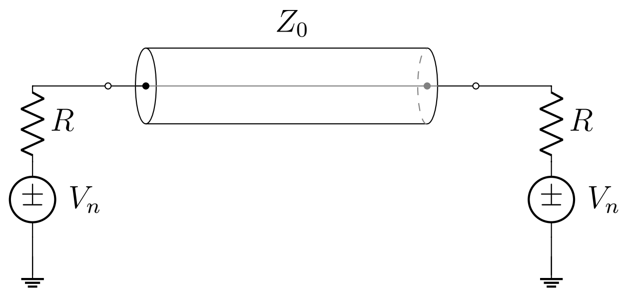

Figure 1.1: Transmission line connecting two resistors.

The thermal agitations due to carriers (electrons or holes) in a conductor above absolute zero temperature can rise to a current to flow, this discovery was first done by J. B. Johnson [2]. Then, H. Nyquist showed the generation of the noise in a resistor is equivalent to the thermal excitations of the energy in the normal modes of electrical oscillation along the transmission line with impedance $Z_0 = R$ connected to two resistors of resistance R (Figure 1.1) [3]. The Power noise generated by a resistor R with a voltage source is represented by $\langle V_n^2 \rangle = 4k_B TRB$ or similarly with a current source by $\langle i_n^2 \rangle = 4k_B TGB$, where k_B is Boltzmann constant, T is the temperature, B is the frequency bandwidth of interest, in Hz, and G is a conductance. The more general formula can be given by

$$\langle V_n^2 \rangle = \frac{1}{2\pi} \frac{4R\hbar\omega d\omega}{e^{\hbar\omega/k_B T} - 1} \tag{1.8}$$

where $\hbar$ is the Planck constant mod 2π. Here, one of the resistors is being replaced by a cavity at temperature T meaning that we require the transmission lines be populated with blackbody radiation and be in thermal equilibrium with the resistors. Since the resistor has a constant temperature T, it absorbs as much power as it emits. Both the emitted and absorbed power per mode are equal to $k_B T B$ in the Rayleigh–Jeans limit.

1.3 Power Spectral Density derivation

There are many ways to get PSD for a noisy system which here I explain one of them, for more information the reader can refer to Appendix C in [1]. In order to calculate PSD for a noisy resistor we can use a noise current source or a noise voltage source as depicted in Figure 1.2.

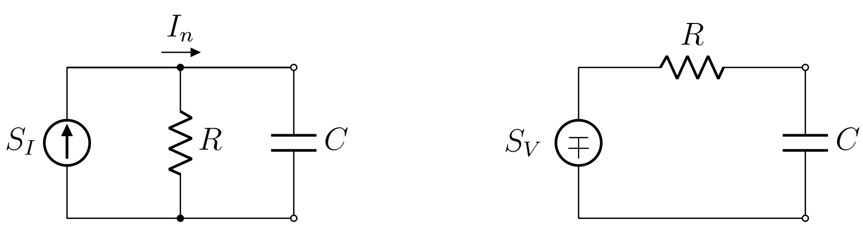

Figure 1.2: Two types of sources, left is the current source with the noise current I_n and the right is the voltage source.

Based on equipartition theorem, each degree of freedom (or mode of oscillation) corresponds to on average an energy of $\frac{1}{2}k_B T$. This energy should be equal to the energy stored in the capacitor which is parallel to the resistor with mean squared noise voltage or power noise as $\langle V_n^2 \rangle = \frac{k_B T}{C}$.

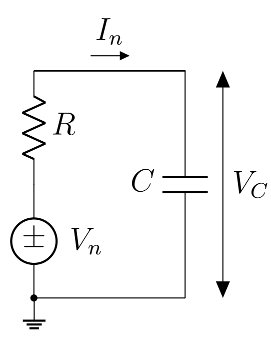

Figure 1.3: RC circuit with a noise source.

In Figure 1.3, the voltage across the capacitor is $V_C = V_n \frac{X_C}{R + X_C}$ where

$X_C = \dfrac{1}{i\omega C}$ and by squaring the two sides of the equation and then taking the mean value of voltage we have

$$\langle V^2 \rangle = \langle V_n^2 \rangle \frac{1}{1 + (\omega RC)^2} \tag{1.9}$$

However, we know from eq. (1.7) that $\langle V_n^2 \rangle = S_V(0) df$ where $S_V(0)$ is the power spectral density measured at very low frequency (near zero) since electrical circuits have low frequency. So we have

$$\frac{k_B T}{C} = \langle V^2 \rangle = \int_0^\infty d|V_C|^2 = \frac{S_V(0)}{2\pi} \int_0^\infty \frac{1}{1 + (\omega RC)^2} d\omega = \frac{S_V(0)}{4RC} \tag{1.10}$$

So that $S_V(0) = 4k_B T R$, and the same approach for noise current source gives us $S_I(0) = 4\dfrac{k_B T}{R}$ or simply $S_I(0) = 4k_B T G$ where G is a conductance. From above discussion we can understand that PSD, S_V, is related to noise power by

$$\langle V_n^2 \rangle = \int_{allowed} \frac{d\omega}{2\pi} S_V(\omega) \tag{1.11}$$

Which evidently explains the bandwidth B introduced above.

1.4 Signal to Noise Ratio (SNR)

One of the most important factor in electronic device engineering is measuring the signal to noise ratio in order to determine how much the desired signal is affected by the background noises. SNR is defined as the ratio of signal power to the noise power [4]

$$SNR = \frac{P_{signal}}{P_{noise}} = \frac{\langle V_s^2 \rangle}{\langle V_n^2 \rangle} \tag{1.12}$$

1.5 The $\mathscr{PT}$ Symmetry

The $\mathscr{PT}$–Symmetric or non–Hermitian systems are in general referred to systems with both parity and time-reversal symmetries together. $\mathscr{PT}$ symmetry was first introduced by Bender [5–8]. Parity symmetry means that the system is invariant under a reflection in spatial coordinate $(x, y, z) \to (-x, -y, -z)$. In quantum mechanics the parity operator $\hat{\mathcal{P}}$ is defines such that $\hat{\mathcal{P}} |x\rangle = |-x\rangle$. Also, the parity operator is a unitary operator meaning that $\hat{\mathcal{P}}^{\dagger} = \hat{\mathcal{P}}^{-1}$, and $\hat{\mathcal{P}}^2 = 1$ which means that $\hat{\mathcal{P}}$ has two eignevalues +1 and -1. The parity operator acts like σ_x, the Pauli matrix ($\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$). In addition, $\left[\hat{\mathcal{P}}, H\right] = 0$ meaning that both have common eigenstates. Time reversal operator $\hat{\mathcal{T}}$ is an anti-unitary operator meaning that $\hat{\mathcal{T}} i \hat{\mathcal{T}}^{-1} = -i$ and inverts the flow of time such that if $\psi(\vec{\mathbf{x}}, t)$ is a vector field then $\hat{\mathcal{T}}\psi(\vec{\mathbf{x}}, t) = \psi'(\vec{\mathbf{x}}, -t)$. Time reversal operator can be represented by a product of unitary matrix and a complex conjugate ($\mathcal{K}$) operators $\mathcal{T} = \mathbf{1}\mathcal{K}$ where $\mathbf{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Time reversal is a very important non-unitary symmetry in physics especially in quantum mechanics when it comes to classifications of phases. In some systems there is a combined $\hat{\mathcal{PT}}$ symmetry which the Hamiltonian is invariant under the action of this symmetry. In general we expect that the Hamiltonian being Hermitian meaning that $H^{\dagger} = H$. The Hermiticity condition of Hamiltonian gives us real (physical) eigenenergies or eigenvalues which can be measured. However, certain complex Hamiltonians are invariant under $\hat{\mathcal{PT}}$ symmetry which are not Hermitian ($H^{\dagger} \neq H$) but they have real and positive eigenvalues in a certain range. So, if H is a non–Hermitian Hamiltonian then $\left[H, \hat{\mathcal{PT}}\right] = 0$.

1.6 The $\mathscr{PT}$–Symmetric RLC circuits

In electronic systems time reversal symmetry changes the sign of resistive impedances since

$$\hat{\mathcal{T}}\{V(t) = I(t)Z(\omega)\} = \{V(-t) = -I(-t)Z(\omega)\} \tag{1.13}$$

Or equivalently $V = -IR$ which means that time reversal changes the sign of resistor $V = I(-R)$. The general Ohm's law $V = IR$ tells about dissipation in the resistor; However, $V = -IR$ represents energy drawing from the resistor.

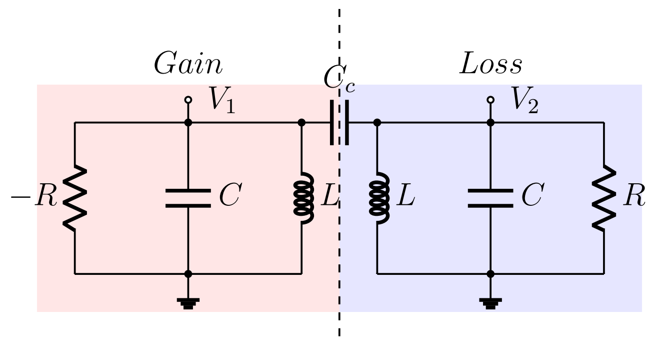

Figure 1.4: $\mathscr{PT}$–Symmetric circuit with loss and gain regions depicted with red and blue colors respectively.

In $\mathscr{PT}$–Symmetric circuits we usually have RLC dimer which are coupled via a capacitor. One RLC equipped with +R (Loss) and the other with -R (Gain) as shown in Figure 1.4. This system is not parity symmetric since if we change the position of RLC circuits with respect to the perpendicular dashed line, the topology of the system will be changed. Also, if time reversal operator acts on the system then the flow of current should be reversed meaning that gain and loss would be interchanged resulting in the change of the topology. But, if $\mathscr{PT}$ acts on the Hamiltonian, first $\mathscr{P}$ changes loss and gain side since parity changes the position with respect to y–axis. However, the action of

$\mathscr{T}$ also changes the loss and gain by changing the current flow so that the system remains unchanged under this operation. It is straightforward to get the Kirchoff's laws for the above circuit.

$$
\begin{aligned}
&i\omega C_c(V_1 - V_2) + i\omega C V_1 + \frac{V_1}{i\omega L} + \frac{V_1}{-R} = 0 \\
&i\omega C_c(V_2 - V_1) + i\omega C V_2 + \frac{V_2}{i\omega L} + \frac{V_2}{R} = 0
\end{aligned} \tag{1.14}
$$

Here we did not consider the mutual inductance between the inductors. The above equation represents a linear homogenous system with two differential equations that can be solved for eigenfrequencies Ω.

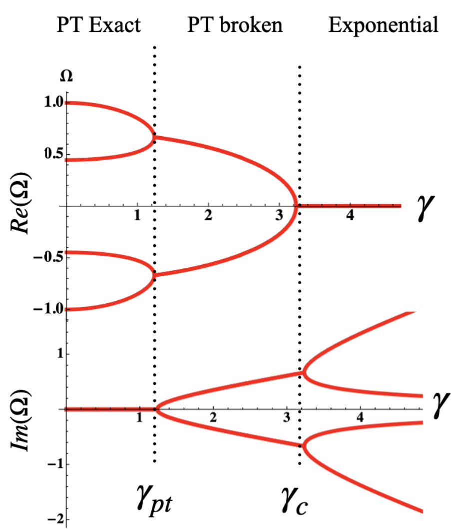

Figure 1.5: The real and imaginary parts of four normal mode frequencies versus γ for $\mu = 0$ and $c = 2$. The exact $\mathscr{PT}$ region contains four real oscillating frequencies. The $\mathscr{PT}$ broken region has exponentially growing and decaying oscillations. The exponential region has completely exponential non-oscillating solutions.

$$
\begin{pmatrix} -\Omega(c+1) + \frac{1}{\Omega} - i\gamma & \Omega c \\ \Omega c & -\Omega(c+1) + \frac{1}{\Omega} + i\gamma \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = 0
$$

Where $\Omega = \frac{\omega}{\omega_0}$ with $\omega_0 = \frac{1}{\sqrt{LC}}$, $\gamma = \frac{1}{R}\sqrt{\frac{L}{C}}$, and $c = \frac{C_c}{C}$ are scaled parameters. The eigenvalues can be obtained [13]

$$\Omega^{\pm}_{\pm} = \pm\frac{\sqrt{\gamma_c^2 - \gamma^2} \pm \sqrt{\gamma_{pt}^2 - \gamma^2}}{2\sqrt{1+2c}} \tag{1.15}$$

Where $\gamma_{pt} = \sqrt{1+2c} - 1$ and $\gamma_c = \sqrt{1+2c} + 1$. There exists four different normal mode eigenfrequencies for this system which both real and imaginary parts of eigenvalues are plotted in Figure 1.5 versus γ. There are three different regions, exact, symmetry broken, and exceptional regions. For $0 < \gamma < \gamma_{PT}$ the regions is exact meaning that there are four pure real eigenfrequencies with no imaginary part which are exactly $\mathcal{PT}$–symmetric. In some texts this region is called unbroken region since $\mathcal{PT}$ symmetry is not broken. In Unbroken region the system is in equilibrium and eigenstates oscillates without decaying or growing [10]. At exactly $\gamma = \gamma_{PT}$, there is a $\mathcal{PT}$ symmetry phase transition and the $\mathcal{PT}$ symmetry is spontaneously broken, this point is called "exceptional point". $\gamma_{PT} < \gamma < \gamma_c$ is "broken region" where $\mathcal{PT}$ symmetry is broken meaning that normal modes are no longer obviously $\mathcal{PT}$-symmetric. In this region the system is not in equilibrium and the eigenvalues exponentially grow and decay. It can be easily seen from the picture that the two opposite imaginary parts lead to a growing and decaying mode, both with the same oscillation frequency. For the broken region, although H and $\mathcal{PT}$ commutes, the eigenstates of the Hamiltonian are no longer simultaneously all eigenstates of $\mathcal{PT}$. In $\gamma < \gamma_c$ region eigenfrequencies are purely imaginary so there are no oscillatory normal modes in the system (over-damped) so that this region is called "exponential region" [9]. The eigenvectors $(V_1, V_2)^T$ associated with the eigenfrequencies $\Omega\pm$ are determined by [13]

$$\begin{pmatrix} V_1 \\ V_2 \end{pmatrix}_{\pm} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -e^{i\phi_\gamma^\pm} \end{pmatrix} \tag{1.16}$$

Where $\phi_\gamma^\pm = \frac{\pi}{2} - \arctan\left(\frac{\Gamma_\pm}{\gamma}\right)$, and $\Gamma_\pm = \Omega_\pm(1+c) - \frac{1}{\Omega_\pm}$. ϕ_γ is the normal mode phase which is the phase of the loss side oscillations relative to the gain side ($\Omega_\pm$ or $-\Omega_\pm$) . When $\gamma = 0$ the normal phase is either 0 or π and when γ increases then the normal modes touch each other at point $\gamma = \gamma_{PT}$. So there are two modes in $0 \le \gamma \le \gamma_{pt}$ region and after this point they coalesce. In Figure 1.6 $\phi_\gamma^+ = 0$ and $\phi_\gamma^- = \pi$ are two phases at $\gamma = 0$, which are purely real eigenmodes exactly before $\gamma = \gamma_{PT}$. The imaginary part of γ_ϕ is shown in Figure 1.7, it is evident that before γ_{PT} there is no imaginary part of eigenmodes; However, after this point there are two parts of imaginary growing eigenmodes.

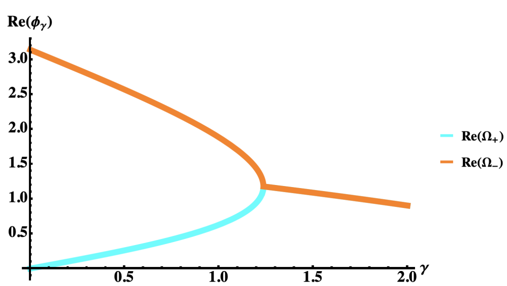

Figure 1.6: The real and imaginary part of SNR versus the phase difference between two current sources.

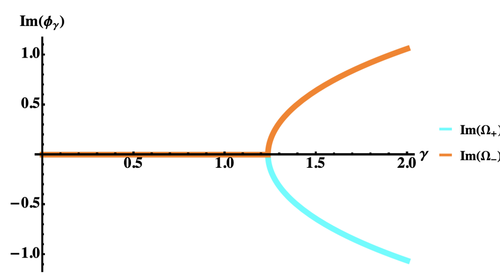

Figure 1.7: Normal mode phase ϕ_γ versus γ for four poles.

1.7 Negative Resistance and the Noise

In an ordinary resistor based on Ohm's law, ($V = RI$), the voltage across the resistor is proportional to the current meaning that the increase in voltage gives rise to the increase in current . However, in a "negative resistor" the increase in voltage across the so-called negative resistor's terminals causes in a drop in current through it. In a negative resistor the power of the signal will be increased in contrast to the ordinary resistor which the power will decay. The negative resistor's circuit is shown in Figure 1.8 containing an amplifying device like 2X op-amp with a positive feedback and a regular resistor. In circuit theory, these circuits are called "Active Resistors". For consideration of noise in this work, the op-amp is considered to be perfect by assuming to be in zero temperature so it is noiseless. Since the op Amp is noiseless, the only noise source is the noise caused by the resistor in the circuit thus the power noise is $\langle V_n^2 \rangle = 4k_BT|R|B$ which R is negative resistance so to avoid confusion with signs we will be careful to include the absolute value.

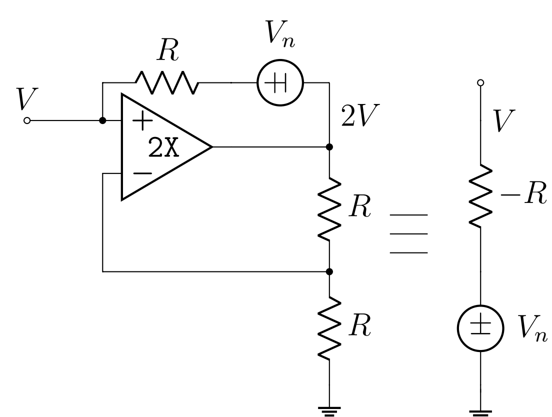

Figure 1.8: Negative resistor. Left picture depicted negative resistor with $2X$ amplifier with a noise source V_n for the lossy resistor which is equivalent to the lossy resistor in the right with the same noise source.

1.8 The Noisy Uncoupled RLC System

For the pedagogical purpose, it is important to first explain an uncoupled noisy non–Hermitian RLC circuit and then comparing it with regenerative RLC circuit where there is only a positive resistance. Since we have negative resistance and positive resistance, we should be careful when we want to calculate $\langle V^2 \rangle$ to use absolute values only where appropriate. In the following, it will be assumed that either R_1 or R_2 could have any numerical sign, and absolute value symbols will be used to correctly manage the algebra.

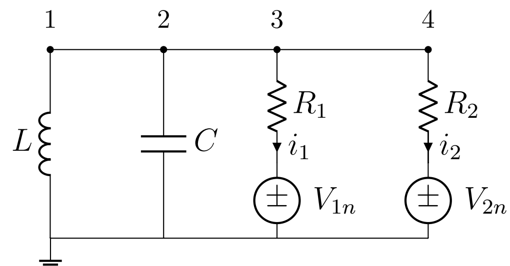

Figure 1.9: Non–Hermitian RLC circuit with loss and gain resistors, R_1 and R_2 respectively.

By imposing the Kirchhoff's law on Figure 1.9 circuit we have:

$$V = \frac{V_{1n}(\frac{R_2}{R_1+R_2}) + V_{2n}(\frac{R_1}{R_1+R_2})}{(1+i(\frac{R_1R_2}{R_1+R_2})(\omega C - \frac{1}{\omega L}))}$$

In order to calculate $\langle VV^* \rangle$

$$\langle V^2 \rangle = \left(\frac{1}{1+\frac{|R_1R_2|^2}{|R_1+R_2|^2}\left(\omega C - \frac{1}{\omega L}\right)^2} \right) \left(\frac{|R_2|^2}{|R_1+R_2|^2} \right) \langle V_{1n}^2 \rangle + \left(\frac{1}{1+\frac{|R_1R_2|^2}{|R_1+R_2|^2}\left(\omega C - \frac{1}{\omega L}\right)^2} \right) \left(\frac{|R_1|^2}{|R_1+R_2|^2} \right) \langle V_{2n}^2 \rangle$$

We know that $S_v(0) = 4k_BRT$ and $\langle V_n^2 \rangle = \frac{1}{2\pi}\int_0^\infty S_v(0)d\omega$ so for the first term above we have :

$$\langle V^2\rangle = \left(|R_1||R_2|^2 + |R_2||R_1|^2\right)\left(\frac{4k_BT}{2\pi(|R_1+R_2|^2)}\right)\int_0^\infty \frac{\omega^2 L^2 d\omega}{\omega^2 L^2 + \frac{|R_1R_2|^2}{|R_1+R_2|^2}\left(\omega^2 LC - 1\right)^2}$$

Then we define u^2 as $\omega^2 LC$ we have, $2LC\omega d\omega = 2udu$ and $\omega = \sqrt{\frac{1}{LC}}u$. Then extract $\dfrac{|R_1R_2|^2}{|R_1+R_2|^2}$ and call $\gamma^2 = \dfrac{L/C}{\dfrac{|R_1R_2|^2}{|R_1+R_2|^2}}$. So we have :

$$\langle V^2\rangle = \left(\frac{k_BT}{C}\right)\left(\frac{\gamma_1+\gamma_2}{\gamma}\right) \tag{1.17}$$

Where, $\gamma_1 = \frac{1}{R_1}\sqrt{\frac{L}{C}}$ and $\gamma_2 = \frac{1}{|R_2|}\sqrt{\frac{L}{C}}$. Here we defined R_2 as a gain and R_1 as a loss.

1.9 The comparison between mean square of voltage in non–Hermitian regenerative RLC system with Gain-Loss RLC circuit

Having discussed the value of V_{ms} now its time to interpret the result by comparing the studied RLC circuit which is considered as a defective (not high quality factor) RLC oscillator, with a perfect oscillator (high quality factor) (see Figure 1.10). We impose further conditions, first $R_1 < R_0$, both R_1 and R_0 are positive, and R_2 is a negative resistor. This will allow a comparison of not only the "regenerative" circuit, where gain is used to compensate for loss, but also compensate for additional loss, similar to what happens in the $\mathscr{PT}$ dimer. We know that for the left figure which is a high quality factor oscillator $\langle V_0^2\rangle = \frac{k_BT}{C}$ and $\gamma_0 = \frac{1}{R_0}\sqrt{\frac{L}{C}}$. Now for the right circuit we put a condition $\frac{1}{R_0} = \frac{1}{R_1} + \frac{1}{R_2}$ so we can substitute $\frac{1}{R_2} = \frac{1}{R_0} - \frac{1}{R_1}$ in eq. (1.17), This imposes

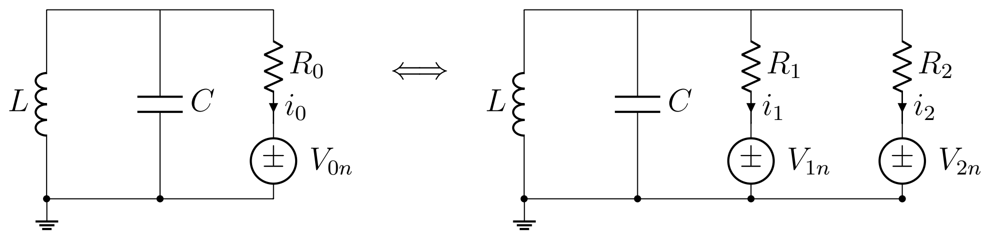

Figure 1.10: The comparison of non–Hermitian.

the operation at (or later close to) threshold. Then we have:

$$\begin{aligned}\langle V^2\rangle &= \left(\frac{k_BT}{C}\right)\left(\frac{\gamma_1+\gamma_2}{\gamma_0}\right)\\ &= \left(\frac{k_BT}{C}\right)\frac{\left(\frac{1}{R_1}+|\frac{1}{R_0}-\frac{1}{R_1}|\right)}{|R_1+\frac{1}{\frac{1}{R_0}-\frac{1}{R_1}}|}\frac{R_1}{\frac{1}{R_0}-\frac{1}{R_1}}\end{aligned} \tag{1.18}$$

After simplifying the equation we have :

$$\langle V^2\rangle = \left(\frac{k_BT}{C}\right)\left(\frac{2R_0}{R_1}-1\right) = \left(\frac{k_BT}{C}\right)\left(\frac{2R_2}{\Delta}-1\right) \tag{1.19}$$

Where $\Delta = R_1 + R_2$. In another words we can write it in terms of γ because $\dfrac{R_0}{R_1} = \dfrac{\gamma_1}{\gamma_0}$ so we have:

$$\langle V^2\rangle = \left(\frac{k_BT}{C}\right)\left(\frac{2\gamma_1}{\gamma_0}-1\right) \tag{1.20}$$

If we consider $\gamma_1 = \gamma_0 + \delta\gamma$ where $\delta\gamma > 0$ then we have:

$$\langle V^2\rangle = \left(\frac{k_BT}{C}\right)\left(1+\frac{2\delta\gamma}{\gamma_0}\right) = \langle V_0^2\rangle + \frac{2\delta\gamma}{\gamma_0} \tag{1.21}$$

This shows that the noise power for the non–Hermitian circuit with gain and loss ($\langle V^2\rangle$)always is bigger than the power noise in a high quality counterpart with only one positive resistor ($\langle V_0^2\rangle$).

Chapter 2

The Noisy non–$\mathscr{PT}$–symmetric and $\mathscr{PT}$–symmetric RLC System

In this chapter both non–$\mathscr{PT}$–symmetric and $\mathscr{PT}$–symmetric RLC systems are introduced in order to get the signal to noise ratio for both cases in two ways, with current source and voltage source.

2.1 Non–Hermitian uncoupled RLC circuit

We start with the general expression describing noise analysis in a single LC resonator with three resistances R_1, R_2, and r representing gain and loss.

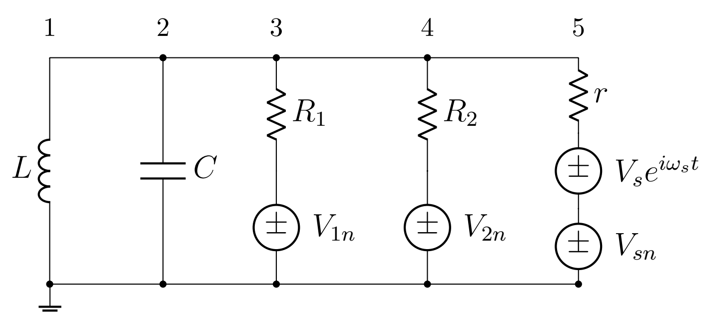

Figure 2.1: Non–Hermitian RLC circuit with positive and negative resistors, and voltage source.

If we consider R_2 as a gain then the parallel combination of R_1 and r should be considered as a loss. Then we describe why near threshold behavior is more simply understood in terms of current sources. It is easy to get the total power noise in Figure 2.1, for the detailed derivation please refer to section B.

$$\begin{aligned}\langle V_{out}^2 \rangle = \left(\frac{k_B T_i}{C}\right)\left(\frac{\gamma_1+\gamma_2}{\gamma}\right) + \left(\frac{k_B T_s}{C}\right)\left(\frac{\gamma_r}{\gamma}\right) \\ + \left(\frac{\langle V_s^2 \rangle \frac{|R_1 R_2|^2}{|r(R_1+R_2)+R_1 R_2|^2}}{1+(\omega_s C - \frac{1}{\omega_s L})^2 (\frac{|r R_1 R_2|^2}{|r(R_1+R_2)+R_1 R_2|^2})}\right)\end{aligned} \tag{2.1}$$

Where $\gamma_1 = \frac{1}{R_1}\sqrt{\frac{L}{C}}$, $\gamma_2 = \frac{1}{|R_2|}\sqrt{\frac{L}{C}}$, $\gamma_r = \frac{1}{|r|}\sqrt{\frac{L}{C}}$ and $\gamma = \frac{|r(R_1+R_2)+R_1R_2|}{|rR_1R_2|}\sqrt{\frac{L}{C}}$. Here in this equation for the last part put $S_v(\omega) = \langle V_s^2 \rangle \delta(\omega - \omega_s)$ and after taking integration we should substitute ω with ω_s. We also assume that (1) the circuit is operated very close to the stability threshold for one of it's modes, and (2) the signal is very close (within Δ) to the mode considered, since the point of this whole project is to explore the best signal-to-noise ratio attainable. Let's consider $r = 0$ and R_1 as a loss and R_2 as a gain. When $r = 0$ the $\gamma = \frac{|R_1+R_2|}{|R_1R_2|}\sqrt{\frac{L}{C}}$, however if we have an ideal gain and loss where $R_1 = -R_2$ then $\gamma = 0$ and the denominator in eq. (2.1) will below up. However, in reality it is hard to have an exact balanced gain and loss, and it is also necessary to allow for a finite band-width signal to be represented by the delta function.

$$\gamma_1 + \gamma_2 = 2\gamma_0 + \Delta$$

Where Δ is an infinitesimal unbalanced real number if $\gamma_1 = \gamma_2 = \gamma_0$. Then the noise mean square voltage in just gain side is:

$$\langle V_n^2 \rangle = \left(\frac{k_B T}{C}\right)\left(\frac{\gamma_2}{\gamma}\right) = \left(\frac{k_B T}{C}\right)\left(\frac{\gamma_2}{\gamma_1+\gamma_2}\right) \approx \left(\frac{k_B T}{C}\right)\left(\frac{\gamma_2}{\Delta}\right) \tag{2.2}$$

Since in the denominator we have the factor R_{II} which is

$$R_{II} = \frac{|rR_1R_2|^2}{|r(R_1 + R_2) + R_1R_2|^2} \tag{2.3}$$

For being $\mathscr{P}\mathscr{T}$–symmetric we should consider

$$\frac{1}{R_2} = -(\frac{1}{R_1} + \frac{1}{r}) \tag{2.4}$$

So that if this condition is satisfied then we have $\frac{1}{R_{II}} \to 0$ and as a consequent R_{II} would blow up. In addition to this, there is a factor $(\omega_s - \omega_0) \to 0$ So we have

$$\begin{aligned} \langle V_s^2 \rangle = \frac{r^{-2}\langle V_s^2 \rangle R_{II}^2}{1 + C^2\omega_s^{-2}(\omega_s + \omega_0)(\omega_s - \omega_0)^2 R_{II}^2} &= \frac{r^{-2}\langle V_s^2 \rangle R_{II}^2}{1 + C^2\omega_s^{-2}(\omega_s + \omega_0)\Delta^2\Delta^{-2}} \\ &\approx \frac{\gamma_r^2\langle V_s^2 \rangle}{\gamma^2} \approx \frac{\gamma_r^2\langle V_s^2 \rangle}{\Delta^2} \end{aligned} \tag{2.5}$$

So SNR is

$$SNR = \frac{\langle V_s^2 \rangle \gamma_r^2 C}{k_B T \gamma_2} \frac{1}{\Delta} \tag{2.6}$$

This shows how the signal to noise ratio ultimately depends on the bandwidth of the signal or equivalently the detector. This is the basis of "regenerative detection", and this should show that although regeneration increases the SNR level, it is not as effective as a non–regenerative resonator where Δ is the inherent resonator loss.

2.2 The Noisy Uncoupled RLC Circuit – Current Noise Source

It is easier to get the eq. (2.6) without the voltage source and just with introducing the current source which is equivalent to the system studied in the previous section. Consider a damped LC circuit or RLC which is driven by two current sources, the equivalent noise I_n, and the "signal" I_s. Then there is only one node with scaled equation of motion. In Figure 2.2, R represents the unbalanced resistor between gain and loss.

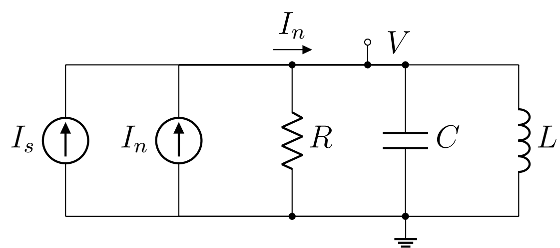

Figure 2.2: Drive noisy non–Hermitian RLC with a current source noise I_s and thermal current I_n. Here R is the sum of two parallel gain and loss resistors.

$$\begin{pmatrix} i\Omega & -1 \\ 1 & i\Omega+\gamma \end{pmatrix}\begin{pmatrix} U \\ V \end{pmatrix} = \begin{pmatrix} 0 \\ U_s+U_n \end{pmatrix} \tag{2.7}$$

Where $\Omega = \omega/\omega_0$, and $U_i = I_i\sqrt{\frac{L}{C}}$, above equation has a solution

$$\begin{pmatrix} U \\ V \end{pmatrix} = \frac{1}{1+i\Omega(\gamma+i\Omega)}\begin{pmatrix} i\Omega+\gamma & -1 \\ 1 & i\Omega \end{pmatrix}\begin{pmatrix} 0 \\ U_s+U_n \end{pmatrix} \tag{2.8}$$

The denominator has two distinct poles $\Omega_+ = \dfrac{i\gamma}{2}+\sqrt{1+\gamma^2/4}$ and $\Omega_- = \dfrac{i\gamma}{2}-\sqrt{1+\gamma^2/4}$. If we assume to get the solution when $\Omega \approx 1$ or $\omega \approx \omega_0$ then the positive pole is $\Omega_+ = 1+\dfrac{i\gamma}{2}$ and the negative pole is $\Omega_- = \dfrac{i\gamma}{2}-\sqrt{1+\gamma^2/4} \approx 2$

so the solution for the node is

$$V = \frac{-i\Omega(U_s + U_n)}{2(\Omega - (1 + \dfrac{i\gamma}{2}))} \tag{2.9}$$

For the RMS signal analysis we should just look at U_s since it is uncorrelated with U_n so that we have to determine the RMS signal at the resonance when $\Omega \approx 1$ the we have

$$\langle V_s^2 \rangle = \frac{\Omega^2 |U_s|^2}{4|\Omega - (1 + \dfrac{i\gamma}{2})|^2} \approx \frac{|U_s|^2}{\gamma^2} \approx \frac{|U_s|^2}{\Delta^2} \tag{2.10}$$

For RMS noise we have

$$\begin{aligned} \langle V_n^2 \rangle &= \frac{1}{2\pi} \int_0^\infty \frac{(L/C) x^2 S_I(\omega) d\omega}{4|x - (1 + \frac{i\gamma}{2})|^2} \\ &= \frac{L S_I(0) \omega_0}{8C\pi} \int_0^\infty \frac{x^2 dx}{4|x - (1 + \frac{i\gamma}{2})|^2} = \frac{k_B T}{C} \frac{\gamma_2}{\gamma} \end{aligned} \tag{2.11}$$

Where here $S_I(0) = \dfrac{4 k_B T}{R}$. Here we used the fact that R is the sum of parallel loss and gain resistors which are R_1 and R_2 respectively. If both gain and loss are balanced, $\gamma = \gamma_1 + \gamma_2 = 0$, but this cannot happen since there is always a small unbalanced factor so that $\gamma_1 + \gamma_2 = \Delta$. Consequently, the signal to noise ratio (SNR)

$$SNR = \frac{|U_s|^2 C}{k_B T \gamma_2} \frac{1}{\Delta} \tag{2.12}$$

Above equation is equivalent to eq. (2.6) if we let $|U_s|^2 = |V_s|^2 \gamma_r^2$. Both equations show that at the end the signal bandwidth dependency of SNR. This is the basis of "regenerative detection", and this should show that although regeneration increases the SNR level, it is not as effective as a non–regenerative resonator where Δ is the inherent resonator loss.

2.3 The Noisy $\mathscr{P}\mathscr{T}$–symmetric RLC Dimer – Current Noise Source

In this section $\mathscr{P}\mathscr{T}$–symmetric RLC Dimer with current sources is studied. The more complicated approach with voltage sources is done in section E.

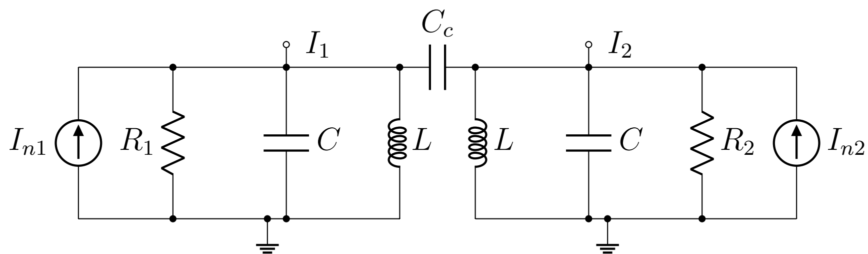

Figure 2.3: $\mathscr{P}\mathscr{T}$–symmetric RLC circuit with two current sources, and two internal noise sources I_{n1} and I_{n2} with respect to loss and gain respectively. C_c is the coupling capacitor which is constant here, and there is no mutual inductance between two inductors.

A $\mathscr{P}\mathscr{T}$ oscillator pair with gain (loss) parameter γ has current sources I_1 and I_2 injected into gain and loss nodes, V_1 and V_2 respectively. Moreover, both R_1 and R_2 have current noise sources I_{n1} and I_{n2} respectively (Figure 2.3). This is a driven, linear system, so that arbitrary currents can be treated as a superposition of different frequency components. Further, if the system is ultimately adjusted to it's $\mathscr{P}\mathscr{T}$-balanced condition, all noise sources and transmission line loading can be accommodated into the choice for final gain and loss parameter and current source components after invoking Norton's equivalence theorem. Assuming the gain and loss come from parallel resistances, the balance of currents leaving each node is expressed in Fourier ($e^{i\omega t}$) form as

$$\begin{pmatrix} i\omega(C+C_c)+\frac{1}{i\omega L}-\frac{1}{R} & -i\omega C_c \\ -i\omega C_c & i\omega(C+C_c)+\frac{1}{i\omega L}+\frac{1}{R} \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \end{pmatrix}$$

$$-\begin{pmatrix} I_1 + I_{n1} \\ I_2 + I_{n2} \end{pmatrix} = 0$$

Then define $\Omega = \frac{\omega}{\omega_0}$ where $\omega_0 = \frac{1}{\sqrt{LC}}$ is a natural frequency of LC circuit, $U_i = I_i\sqrt{\frac{L}{C}}$, $\gamma = \dfrac{1}{R}\sqrt{\frac{L}{C}}$, and $c = \dfrac{C_c}{C}$. Then the solution is

$$\begin{pmatrix} V_1 \\ V_2 \end{pmatrix}_{\pm} = \frac{1}{(\Omega^2 c^2 - \Gamma^2 - \gamma^2)} \begin{pmatrix} i\Gamma + \gamma & -i\Omega c \\ -i\Omega c & i\Gamma - \gamma \end{pmatrix} \begin{pmatrix} U_1 + U_{n1} \\ U_2 + U_{n2} \end{pmatrix} \tag{2.13}$$

Where $\Gamma = \Omega(1 + c) - \frac{1}{\Omega}$. The denominator can be factored using the explicit solutions to the PT-symmetric eigenvalue problem solved by its roots,

$$\Omega^2 c^2 - \Gamma^2 - \gamma^2 = -\frac{1 + 2c}{\Omega^2}(\Omega^2 - \Omega_+^2)(\Omega^2 - \Omega_-^2) \tag{2.14}$$

Where $\Omega_\pm = \dfrac{\sqrt{\gamma_c^2 - \gamma^2} \pm \sqrt{\gamma_{pt}^2 - \gamma^2}}{2\sqrt{1 + 2c}}$ with $\gamma_{pt} = \left|\sqrt{1 + 2c} - 1\right|$ and $\gamma_c = \sqrt{1 + 2c} + 1$. Also we have two other poles which are $\Omega^+ = -\Omega_+$ and $\Omega^- = -\Omega_-$, they are considered in the equation eq. (1.15).

$$\begin{pmatrix} V_1 \\ V_2 \end{pmatrix}_{\pm} = \frac{-\Omega^2 \begin{pmatrix} (i\Gamma + \gamma)(U_1 + U_{n1}) + i\Omega c(U_2 + U_{n2}) \\ i\Omega c(U_1 + U_{n1}) + (i\Gamma - \gamma)(U_2 + U_{n2}) \end{pmatrix}}{(1 + 2c)(\Omega^2 - \Omega_+^2)(\Omega^2 - \Omega_-^2)} \tag{2.15}$$

When the expression is evaluated near either of the resonance frequencies, further simplifi-cation is possible. Let $\Omega - \Omega_0$ be small, where Ω_0 is either Ω_+ or Ω_-. Factoring the i from the current vector and expressing the current coefficients in polar form gives

$$\begin{pmatrix} (i\Gamma + \gamma)(U_1 + U_{n1}) + i\Omega c(U_2 + U_{n2}) \\ i\Omega c(U_1 + U_{n1}) + (i\Gamma - \gamma)(U_2 + U_{n2}) \end{pmatrix} \approx i\Omega_0 c \begin{pmatrix} \frac{i\Gamma + \gamma}{\sqrt{\Gamma^2 + \gamma^2}}(U_1 + U_{n1}) + (U_2 + U_{n2}) \\ (U_1 + U_{n1}) + \frac{i\Gamma - \gamma}{\sqrt{\Gamma^2 + \gamma^2}}(U_2 + U_{n2}) \end{pmatrix} \tag{2.16}$$

Where $\Omega_0 c \approx \sqrt{\Gamma^2+\gamma^2}$. Also we can define the argument as $\frac{i\Gamma+\gamma}{\sqrt{\Gamma^2+\gamma^2}} = e^{i\phi_\gamma}$ such that $\phi_\gamma = \frac{\pi}{2} - \arctan\left(\frac{\Gamma}{\gamma}\right)$ which we call "normal mode phase". Remarkably, the resonantly-driven PT dimer (in the exact phase) has voltage oscillations with equal magnitudes regardless of any difference in the applied amplitudes. This simply confirms the "rigidity" of normal modes driven on-resonance. The relative phase of the voltages, ϕ_γ, is that of the normal mode

$$\begin{pmatrix} V_1 \\ V_2 \end{pmatrix}_{\pm} = \frac{-ic \begin{pmatrix} e^{i\phi_\gamma}(U_1+U_{n1}) + (U_2+U_{n2}) \\ (U_1+U_{n1}) + e^{-i\phi_\gamma}(U_2+U_{n2}) \end{pmatrix}}{2(1+2c)(\Omega-\Omega_\pm)\left(1-(\frac{\Omega_\mp}{\Omega_\pm})^2\right)} \tag{2.17}$$

For Measuring the RMS voltage and noise voltage in the system we need to square the voltage nodes which tell us it does not matter which one we choose.

$$|V|^2 = \frac{c^2(U_{n1}^2+U_{n2}^2)}{4(1+2c)^2|\Omega-\Omega_\pm|^2\left[1-(\frac{\Omega_\mp}{\Omega_\pm})^2\right]^2} \tag{2.18}$$

Here U_1 and U_2 represent the broad-band current noise sources which are near resonance, and can be interpreted as the residues corresponding to each of the poles associated with the filtering transfer-function. In general $\langle I_n^2\rangle = \frac{4k_B T\Delta\omega}{R} = 2\pi S_I(\omega)\Delta\omega$, where $\Delta\omega = B$ is the bandwidth and S_I is the spectral density of current fluctuations of the random current source. Here we define $\eta^2 = \frac{L}{C}\langle I_n^2\rangle$. In reality the system hardly it is on resonance and there exists a detuning Δ such that moves the poles from the real axis to the complex plane. Experimentally, the de-tuning is absolutely necessary: maintaining a real system exactly at threshold is impractical. Even the slightest fluctuations through the threshold condition will dramatically destroy the advantage of the regenerative bandwidth narrowing due to the corresponding phase fluctuations.

$$|V_n|^2 = \frac{c^2(\eta_1^2+\eta_2^2)}{4(1+2c)^2|\Omega-\Omega_\pm+i\Delta|^2\left[1-(\frac{\Omega_\mp}{\Omega_\pm})^2\right]^2} \tag{2.19}$$

If we assume that both noise current source are in the same temperature and they have the same bandwidth (B) then they are identical. If we compute the system on resonance so we have:

$$|V_n|^2 = \frac{c^2\frac{L}{C}(\frac{4k_BTB}{R}+\frac{4k_BTB}{|-R|})}{4(1+2c)^2\Delta^2\left[1-(\frac{\Omega_\mp}{\Omega_\pm})^2\right]^2} = \frac{2(\gamma_R+\gamma_{-R})k_BTB\sqrt{\frac{L}{C}}}{(1+2c)^2\Delta^2\left[1-(\frac{\Omega_\mp}{\Omega_\pm})^2\right]^2} \tag{2.20}$$

In order to determine the noise source RMS we should determine the residue. We know that

$$|V_n|^2 = \frac{Lc^2}{4C(1+2c)^2}\left\{\frac{\omega_0}{2\pi}\int_0^\infty \frac{d\Omega\left(S_1^I(0)+S_2^I(0)\right)}{|\Omega-\Omega_\pm+i\Delta|^2\left[1-(\frac{\Omega_\mp}{\Omega_\pm})^2\right]^2}\right\} \tag{2.21}$$

Since we have $|\Omega^2-\Omega_\pm^2| = (\Omega-\Omega_\pm)(\Omega-\Omega_\pm^*)$ and the remaining term in the denominator is $i\Delta$. We have

$$\begin{aligned}|V_n|^2 &= \frac{Lc^2\omega_0(\left(S_1^I(0)+S_2^I(0)\right)}{8C(1+2c)^2\Delta}\left(\left[1-(\frac{\Omega_\mp}{\Omega_\pm})^2\right]^{-2}+\left[1-(\frac{\Omega_\pm}{\Omega_\pm})^2\right]^{-2}\right)\\ &= \frac{k_BT}{C}\left(\frac{c^2\left(\gamma_R+\gamma_{-R}\right)}{2(1+2c)^2\Delta}\right)\left(\left[1-(\frac{\Omega_\mp}{\Omega_\pm})^2\right]^{-2}+\left[1-(\frac{\Omega_\pm}{\Omega_\mp})^2\right]^{-2}\right)\end{aligned} \tag{2.22}$$

Now for the source part, we assumed that first the noise sources are uncorrelated so they do not talk to each other or feel the presence of one another. However, the source signals could be correlated, described by a phase difference in the frequency domain. Let $I_1 = I_s$ and $I_2 = I_se^{i\theta}$ where θ is the phase difference between two current sources. So for the voltage node we have (ignore the noise sources since they do not talk to current sources)

$$\begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \frac{-ic \begin{pmatrix} e^{i\phi_\gamma} U_s + e^{i\theta} U_s \\ U_s + e^{-i\phi_\gamma} e^{i\theta} U_s \end{pmatrix}}{2(1+2c)(\Omega - \Omega_\pm)\left(1 - (\frac{\Omega_\mp}{\Omega_\pm})^2\right)} \tag{2.23}$$

Then with a simple calculation we have

$$e^{i\phi_\gamma} U_s + e^{i\theta} U_s = U_s e^{i\theta}(e^{i(\phi_\gamma - \theta)} + 1) = U_s e^{i\theta} e^{i(\phi_\gamma - \theta)/2} \cos(\phi_\gamma - \theta)$$

$$|V_s|^2 = \frac{c^2 |U_s|^2 \cos^2(\phi_\gamma - \theta)}{4(1+2c)^2 |\Omega - \Omega_\pm + i\Delta|^2 \left[1 - (\frac{\Omega_\mp}{\Omega_\pm})^2\right]^2} \tag{2.24}$$

Where $U_s = I_s\sqrt{\frac{L}{C}}$. And for all four poles we have and if $\Omega \approx \Omega_\pm$

$$|V_s|^2 = \frac{c^2 |U_s|^2 \cos^2(\phi_\gamma - \theta)}{4(1+2c)^2 \Delta^2} \left(\left[1 - (\frac{\Omega_\mp}{\Omega_\pm})^2\right]^{-2} + \left[1 - (\frac{\Omega_\pm}{\Omega_\mp})^2\right]^{-2} \right) \tag{2.25}$$

It is evident that here there is a dependency of RMS voltage source on the difference between two phases, ϕ_γ and θ in comparison to noise RMS. In order to maximize the output we should maximize the $\cos^2(\phi_\gamma - \theta)$ which gives

$$\phi_\gamma - \theta = n\pi \qquad n \in \mathbb{Z} \tag{2.26}$$

If $n = 0$ then $\phi_\gamma = \theta$, meaning that the two phases should match to give the maximum output. In other words $\theta = \arctan\left(\frac{\Gamma}{\gamma}\right) - \frac{\pi}{2}$ up to $n\pi$ factor. Then we have for signal to noise ratio for

$$SNR = \frac{C|U_s|^2}{2k_B T \gamma_2} \frac{1}{\Delta} \tag{2.27}$$

Where $\gamma_2 = \gamma_{-R}$. Note that the signal amplitude and the noise power have a dependency with a $\frac{1}{\Delta^2}$ and $\frac{1}{\Delta}$ de-tuning singularity respectively, so the signal-to-noise ratio grows as $\frac{1}{\Delta}$ as expected for a simple regenerative bandwidth narrowing, in spite of both being amplified by the regeneration (gain). This enhancement will be theoretically limited to the inherent bandwidth of the signal's information content. Moreover, this ratio is as half as in eq. (2.12) this is because of the gain side noise which adds more noise to the system. Another way to get the more precise SNR is derived in E

$$|V_s|^2 = \frac{2C\gamma_2 V_s^2 \gamma_r^2}{k_B T \gamma^2 c^2} \frac{1}{\Delta} \times \frac{\left(\Gamma^2 + \gamma^2 + \Omega_+^2 c^2 + 2\Omega_+ c(\Gamma \cos\phi + \gamma \sin\phi)\right)}{4|\Omega_+|^2} \tag{2.28}$$

In above equation we just consider the exact region. SNR versus θ is shown in Figure 2.4 based on eq. (2.28). It is easy to see that there is a maximum near -1.08 where $\phi_\gamma = 1.08$ with $c = 2$, and $\gamma = 1.1$.

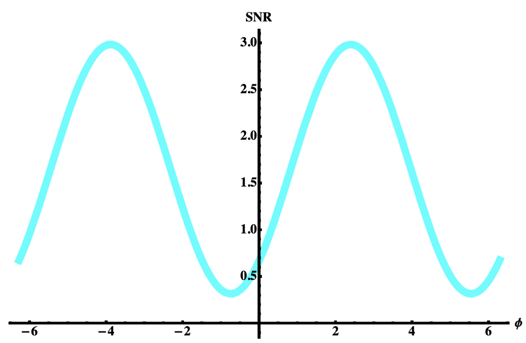

Figure 2.4: SNR versus the phase difference between two current sources.

2.4 Conclusion

By comparing eq. (2.12) and eq. (2.27), it is evident that non–Hermitian non–$\mathscr{P}\mathscr{T}$–symmetric and $\mathscr{P}\mathscr{T}$–symmetric are different only in the factor 2, otherwise the result it the same. The advantage of having $\mathscr{P}\mathscr{T}$–symmetric over non–$\mathscr{P}\mathscr{T}$–symmetric is that the phase difference between two external current sources is tunable. When it comes to signal to noise ratio (SNR) there is a disadvantage of having $\mathscr{PT}$–symmetric compare to the non–$\mathscr{PT}$–symmetric. The effect of noise in $\mathscr{PT}$–symmetric circuits has been recently studied since in reality the noise should be considered and it cannot be avoided. Noise analysis of $\mathscr{PT}$–symmetric systems would allow us to understand the effectiveness of gain and loss systems which are useful for reducing the loss. However, the noise in $\mathscr{PT}$–symmetric circuits cannot be neglected because of thermal (Johnson-Nyquist), shot, flicker or even quantum noise in systems. Studies have shown that although non–noisy $\mathscr{PT}$–symmetric circuits can have higher quality factor; However, if the effect of noise is considered then the $\mathscr{PT}$–symmetric systems would prone to a high noise figure [11]. However, the authors of paper [12] claim that the suggested $\mathscr{PT}$–symmetric circuit can filter out high-frequency thermal noise. Nonetheless, it is unclear in the paper what circuit represents the equations since the circuit is not on a threshold. The present study will shed light on the noise concept in $\mathscr{PT}$–symmetric circuit in a rigorous way.

Chapter 3

Parametric Amplification and Noisy Circuits

In chapter 2, the effect of noise in $\mathscr{PT}$–symmetric systems has been discussed. Now we aim to test the feasibility of noise power pumping based on parametric amplification. This work involves the theoretical investigation of an LC resonator connected to a transmission line where the resonator capacitance is cyclically modulated. The goal is to reduce the noise in a cold bath by operating a Carnot cycle between hot noisy bath and cold less-noisy bath. The experimental part of this project will the basis for future work.

3.1 Parametric Amplification in LC circuit

Parametric amplifiers were developed to lower the noise in electronic systems where one of the circuits' parameters, like capacitance, varies with time (nonlinear circuit parameter). Time-varying parameter can provide frequency mixing between two frequencies known as "idler" and "signal" where the higher frequency is always signal and the other is idler. This mixing frequency can pump energy into the system and there is a pumping frequency (ω_p) which is the sum of the signal frequency (ω_s) and idler frequency (ω_i).

$$\omega_p = \omega_s + \omega_i \tag{3.1}$$

There are two modes in the system which the coupling between the modes in such a system is called *parametric coupling* [24]. The mechanism of pumping is exactly similar to the child on a swing where the child uses the center of the gravity to pump the swing with the frequency which is twice the swing natural frequency. Therefore at each cycle, the energy is pumped into the swing to make the amplitude larger. The same concept is applied to the time-varying capacitance in a LC circuit. Consider a single LC circuit which we can mechanically push and pull the capacitance's plates in order to change the capacitance of C. Let $C(t) = C_0 + \Delta C \cos(\omega_p t + \phi)$ be the equation for this model where C_0 is the un-modulated capacitance, and $\Delta C \cos(\omega_p t + \phi)$ represents the variation of the capacitance with respect to time.

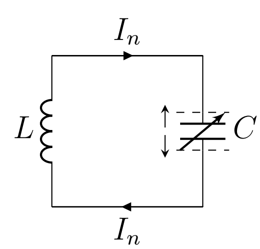

Figure 3.1: Parametric amplification in LC circuit. The capacitance is parametrically changed via time $C(t) = \Delta C \cos(\omega_p t + \phi)$.

When the capacitor is fully charged we have to do a work while pulling the plates apart since the plates have opposite charges and they tend to attract to each other. This work will increase the energy in the electric field between the capacitor's plates. Moreover according to $C = \dfrac{\epsilon A}{d}$ where d is the distance between two plates and A is the plates' area, we are reducing the capacitance by increasing the distance between the plates. Because it takes time that charges to to be changed by decreasing C, so from $v = \frac{Q}{C}$ we can infer that the magnitude of v is increased, regardless of the sign, as you can see in Figure 3.2 that with the increase of d, the voltage v has an instantaneous jump resulting in the increase in the amplitude. After a quarter of resonance we don't have any charges on the plates so no force is required to put the plate in the initial position. Another quarter when the plates are charged with the opposite sign, by pulling apart more energy will be added to the circuit. Energy at twice the resonance frequency can be pumped into the circuit and the oscillation amplitude will grow. So energy is fed into the system. This is an example of "degenerate parametric oscillator" meaning that both signal and idler frequency is the same and both are equal to half of the pump frequency (Exactly like the child on swing example above) [25].

$$\omega_{idler} = \omega_{signal} = \frac{1}{2}\omega_p \tag{3.2}$$

Here it means that you have to push them back at twice natural frequency then you can fed energy into your circuit at each cycle. Pull apart when it is fully charged and pull together when the voltage is zero! Because of the phase, if the phase is $\phi/2$ then when the plates are fully charges and push together then the circuit will give up energy and the circuit oscillation will damped out rather than growing. Here the variation of capacitance instead of being a square wave is sinusoidal wave as depicted in the bottom of Figure 3.2.

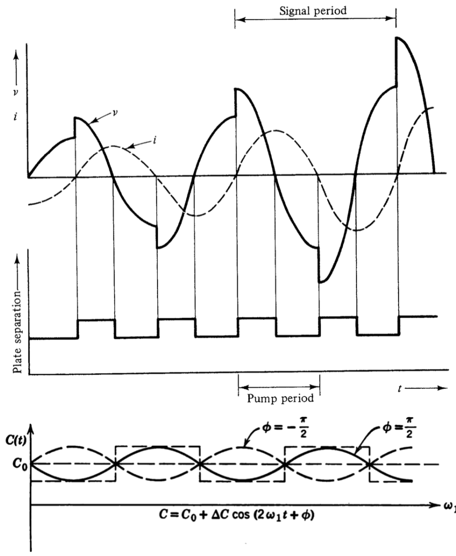

Figure 3.2: Degenerate parametric oscillator [24, 25].

3.2 Degenerate Parametric Amplifier

In order to study couple mode theory of the above system consider LC circuit with $C(t) = C_0 + \Delta C \cos(\omega_p t + \phi) = C_0 + C_p(t)$ where $\omega_p = 2\omega_1$ which ω_1 is the natural frequency for C_1 and C_0 in parallel called C_{11} (section 3.2).

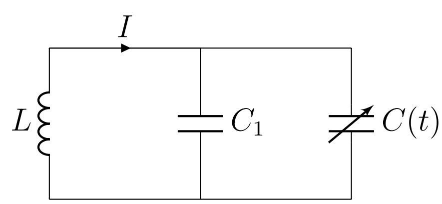

Figure 3.3: Degenerate parametric LC circuit with time varying parallel plate capacitor $C(t)$.

It is easy to wrote the couple mode equations as follows

$$\frac{da}{dt} = -i\omega_1 a + \frac{d}{dt}\left(\frac{C_p}{2C_{11}}(a - a^*)\right)$$
$$\frac{da^*}{dt} = +i\omega_1 a^* - \frac{d}{dt}\left(\frac{C_p}{2C_{11}}(a - a^*)\right) \tag{3.3}$$

If $C_p \to 0$ then there is no coupling between the normal modes a and a^*. But here as $C_p \neq 0$ so that the pump couples a and a^*-modes. For simplifying the equations we consider weak coupling between modes ($\Delta C \ll C_{11}$). Also take the anzats $a(t) = A(t)e^{-i\omega_1 t}$ where is $A(t)$ a slowly varying function of time, and put it in the equation above with some other simplifications (for more details please refer [24])

$$\begin{aligned}\frac{da}{dt} &= -i\omega_1 a + c_{12}e^{-2i\omega_1 t}a^* \\ \frac{da^*}{dt} &= +\omega_1 a^* + c_{21}e^{2i\omega_1 t}a\end{aligned} \tag{3.4}$$

Where $c_{12} = -i\omega_1 \dfrac{\Delta C}{4C_{11}}e^{i\phi} = c_{21}^*$. The complete solution of the problem for the pump phase chosen to be $\phi = \pi/2$ is

$$\begin{aligned}a(t) &= -\frac{i}{2}\sqrt{C_{11}}V_m e^{-i\omega_q t}e^{|s|t} \\ a^*(t) &= \frac{i}{2}\sqrt{C_{11}}V_m e^{i\omega_q t}e^{|s|t}\end{aligned} \tag{3.5}$$

Where $s = \pm\omega_1 \dfrac{\Delta C}{4C_{11}}$ and for $\phi = -\pi/2$ we have the same equations with $-|s|$. Figure 3.2 shows pumping in these two phases which the energy s extracted from the circuit.

3.3 Carnot engine of parametrically pumped LC between two transmission lines with different temperatures

In this section we aim to explain the Carnot engine with parametrically amplified LC resonator between two thermal transmission lines. Here we use a noise source which is random so it cannot be manipulated synchronously. In

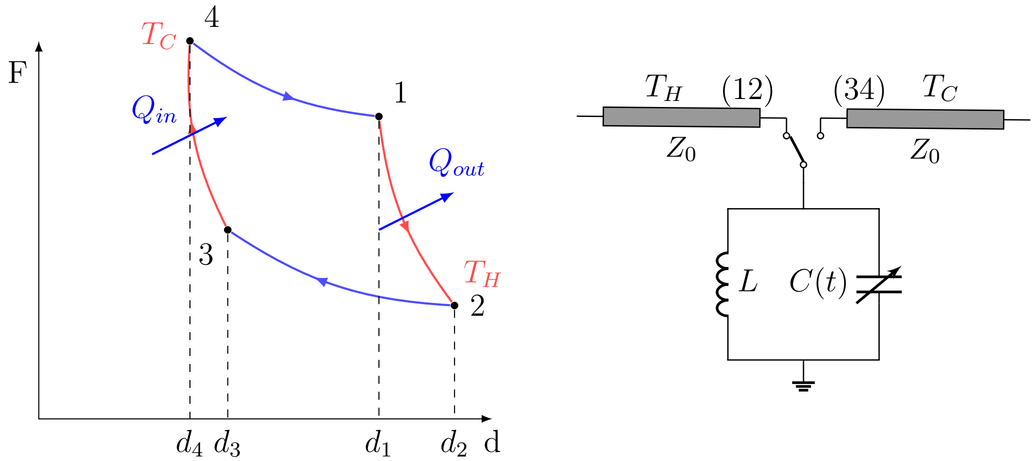

Figure 3.4: Left picture: F-d diagram of Carnot cycle, where blue and red lines depict isothermal and adiabatic paths respectively. **Right picture**: shows the suggested LC resonator with parametrically amplified capacitance between two thermally difference transmission lone with noise sources.

other words, we do not have a degenerate parametric oscillator, instead we have *non-degenerate* parametric oscillator. As a result, this system cannot modulate or have a pump frequency at twice the natural frequency of LC oscilator. A slow modulation can be seen to act on the RMS voltage because of the work done against the RMS force, and also there is an exact solution that is not restricted to small modulations and can be used to understand the "pumping" of the noise. Here the LC resonator is the Carnot engine and the work supplied by the modulator mechanism, and the baths are transmission lines in two different temperatures T_H and T_C, hot and cold baths respectively. Also, an irreversible isothermal process means connecting to one of the noise baths (thermal transmission lines), and an irreversible adiabatic process means disconnecting the oscillator from baths The Carnot mechanism is as follows which is shown in the left side of Figure 3.4, all processes are reversible.

- **1 → 2 (Isothermal)** By pulling apart the capacitor's plates the distance between two plates (d) will increase so that C decreases. As a result, the work is done on LC resonator since the capacitor's plates are inclined to attract each other. So that this work goes into the transmission line as heat (Q_{out}), and it leaves the LC resonator. Note that the expansion of C adds energy to the isothermal baths unlike the ideal gas case. Based on the change of work $\Delta U = Q_{out} - W_{12}$ we have $Q_{out} = W_{12}$, so the system (LC resonator) gives the energy to the environment (transmission line) from the warm bath. For computing the work done on the system we have to calculate the force by using $\langle V^2 \rangle = \frac{k_B T_H}{C}$ where T_H is the temperature at high, and $C = \dfrac{\epsilon A}{d}$

$$F = \frac{q^2}{2\epsilon A} \Longrightarrow \langle F \rangle = \frac{C^2 \langle V^2 \rangle}{2\epsilon A} = \frac{C^2 \frac{k_B T_H}{C}}{2\epsilon A} = \frac{1}{2}\frac{k_B T_H}{d} \tag{3.6}$$

For calculating the work it is sufficient to take $W = \int F dx$ so we have

$$\Delta W = \frac{k_B T_H}{2} \ln\left(\frac{d_1}{d_2}\right) = W_{12} = Q_{out} = \frac{k_B T_H}{2} \ln\left(\frac{d_0}{d_2}\right) \tag{3.7}$$

Where $d_1 = d_0$ and d_2 are the initial and final plates' distance respectively.

- **2 → 3 (Adiabatic)** Isolate the system by disconnecting the LC resonator from two baths so that $\Delta U = \Delta Q + \Delta W = \Delta W$ ($\Delta Q = 0$). In this case let the capacitance plates increase so that the work is done on the LC resonator. For calculating the charge on the plates consider the LC resonator in the right side of Figure 3.4 which the exponentially time-varying gap is $d(t) = d_0 e^{\alpha t}$ where d_0 is the gap at the initial time ($t = 0$), and α is a constant and defines the rate of exponentially change.

From this relation we can have

$$C(t) = \frac{\epsilon A}{d(t)} = C_0 e^{-\alpha t} = \frac{\epsilon A}{d_0} e^{-\alpha t} \tag{3.8}$$

By using Kirchhoff's law we have

$$V_L + V_C = 0 \Longrightarrow L\frac{d^2q}{dt^2} + \frac{q}{C_0} e^{\alpha t} = 0 \tag{3.9}$$

Then we have $\dfrac{d^2q}{d\phi^2} + e^{\zeta\phi} q = 0$ where $\phi = \dfrac{t}{\sqrt{LC_0}}$ and $\zeta = \alpha\sqrt{LC_0}$. The solution to this equation is

$$q[\zeta] = c_1 J_0(\frac{2\sqrt{e^{\zeta\phi}}}{\zeta}) + c_2 n_0(\frac{2\sqrt{e^{\zeta\phi}}}{\zeta}) \tag{3.10}$$

Where J_0 and n_0 are Bessel function and Neumann function respectively. For simplifying the solution it is better to use approximation

$$q(t) = c_1 \sqrt{\frac{\zeta}{\pi\sqrt{e^{\zeta\phi}}}} \cos\left(\frac{2\sqrt{e^{\zeta\phi}}}{\zeta} - \frac{\pi}{4}\right) \tag{3.11}$$

Replacing back ζ and ϕ, so we have

$$q(t) = c_1 \sqrt{\frac{\alpha\sqrt{LC_0}}{\pi} e^{-1/4\alpha t}} \cos\left(\frac{2\sqrt{e^{\alpha t}}}{\sqrt{LC_0}\alpha} - \frac{\pi}{4}\right) \tag{3.12}$$

Call $\dfrac{2\sqrt{e^{\alpha t}}}{\sqrt{LC_0}\alpha} - \dfrac{\pi}{4} = \gamma$ and calculate the voltage by $q(t) = C(t)V$ then we have for $c_1 = 1$

$$V^2 = \left(\frac{\alpha}{\pi}\right) \sqrt{\frac{L}{C_0}} e^{(3/2)\alpha t} \cos^2\gamma \tag{3.13}$$

By taking the average:

$$\langle V^2 \rangle = \left(\frac{\alpha}{2\pi}\right) \sqrt{\frac{L}{C_0}} e^{(3/2)\alpha t} \tag{3.14}$$

So we can calculate the adiabatic force

$$\langle F \rangle = F_0 \sqrt{\frac{d_0}{d}} \tag{3.15}$$

Where $F_0 = \frac{\alpha}{4\pi d_0^2}\sqrt{\frac{L}{C_0}}$. In order to calculate the work done on the environment one should calculate $\Delta W = \int F dx$, and bear in mind that here $d_0 = d_2$

$$\Delta W = 2F_0 d_0 (\sqrt{\frac{d_3}{d_0}} - 1) \tag{3.16}$$

- **3 → 4 (Isothermal)** Connecting system to the cold bath (path (34)). Allow the plates to come back together, here the work is done by the LC resonator, instead there is heat going into the LC resonator called Q_{in}, and $\Delta U = Q_{in} - W_{23} = 0$ then $Q_{in} = W_{23}$.

$$\Delta W = \frac{k_B T_C}{2} \ln (\frac{d_3}{d_4}) = Q_{in} = \frac{k_B T_C}{2} \ln (\frac{d_0}{d_4}) \tag{3.17}$$

Where $d_3 = d_0$.

- **4 → 1 (Adiabatic)** Isolate the system by disconnecting the LC resonator from two baths so that $\Delta U = \Delta Q + \Delta W = \Delta W$, where $\Delta Q = 0$ and the work is done on the LC resonator.

$$\Delta W = 2F_0 d_0 (\sqrt{\frac{d_1}{d_0}} - 1) \tag{3.18}$$

In order to get the efficiency of the Carnot engine we know that $W = Q_{out} - Q_{in}$ so we have

$$\eta_{\text{Carnot}} = \frac{W}{Q_{out}} = \frac{Q_{out} - Q_{in}}{Q_{out}} \tag{3.19}$$

But since $\frac{Q_{out}}{T_H} = \frac{Q_{in}}{T_c}$ so we have

$$\eta_{\text{Carnot}} = 1 - \frac{T_C}{T_H} \tag{3.20}$$

However in our case it is different from above equation. Consider the Carnot cycle in Figure 3.4 , for calculating the work we need to get the relationship between distances. With an easy calculation we can get

$$\frac{T_C}{T_H} = \sqrt{\frac{d_3}{d_2}} = \sqrt{\frac{d_1}{d_4}} = \frac{d_2}{d_4} \tag{3.21}$$

It is easy to get the work which is approximately

$$\begin{aligned} W = W_{(41)} - W_{(23)} &= \frac{\alpha}{2\pi}\sqrt{\frac{L}{C_0}}\left\{\frac{1}{d_4}(\sqrt{\frac{d_1}{d_4}} - 1) - \frac{1}{d_2}(\sqrt{\frac{d_3}{d_2}} - 1)\right\} \\ &= \frac{\alpha}{2\pi d_2}\sqrt{\frac{L}{C_0}}\left\{\frac{T_C}{T_H}(\frac{T_C}{T_H} - 1) - (\frac{T_C}{T_H} - 1)\right\} \end{aligned} \tag{3.22}$$

At the end, for calculating the Carnot efficiency we have

$$\eta_{\text{Carnot}} = \frac{W}{Q_{out}} = \frac{\alpha\sqrt{L}}{k_B\sqrt{C_0}}\frac{1}{d_2\ln\left(\frac{d_1}{d_2}\right)}\left\{\frac{T_C}{T_H^2}(\frac{T_C}{T_H} - 1) - (\frac{T_C}{T_H} - 1)\right\} \tag{3.23}$$

Here we can see the dependency of Carnot engine efficiency on cold and hot reservoirs' temperatures. Based on the derivation for the work, Carnot engine can be possible for pumping the energy from hot and cold transmission lines, with the work calculated earlier we can pump energy from hot bath to cold bath based on parametric amplification concept.

3.4 Varactor Diode, a Candidate for Varying Capacitor

The best candidate for time varying capacitance is a varactor diode. The internal capacitance of diode varies with the change of the reverse voltage so that the capacitance is tunable. A diode is a semiconductor devise comprising a p–n junction. In a p–n junction there are three main regions, p, n, and a depletion region. The p- and n-type semiconductors have the majority positive and negative charge carriers respectively and are good conductors. The depletion region forms by diffusive annihilation of carriers and is insulating. It becomes larger when we have a reverse bias applied to the p-n junction when the positive voltage terminal is attached to n-type and negative voltage terminal attaches to the p-type semiconductor. In result there are more holes in P–type attracting to the negative terminal and more free electrons in N–type going towards the positive terminal and leaving behind negative and positive charge carriers in depletion region respectively. As a result, the depletion region gets wider with applied reverse bias voltage.

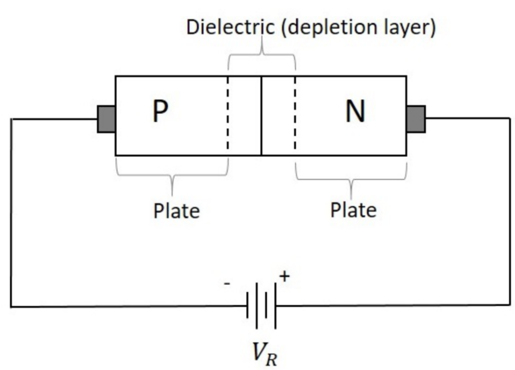

Figure 3.5: p-n junction diode.

In Figure 3.5 the insulating depletion region acts as a capacitor, and p and n-type semiconductors act as the conducting capacitor plates forming the parallel plate capacitor. With the increase of the inverse biased the depletion region widens leading to the decrease of the capacitance, and when the reverse biased voltage decreases the depletion region narrows and the capacitance would increase since $C = \dfrac{\epsilon A}{d}$.

3.5 Experimental Setup for Noise Power Pump LC resonator

In designing the circuit containing the varactor diode which should always be in the reverse biased we have to use the back-to-back operation because of the fact that RF tuning may cause the diode to work in forward biased. The operation is that two diodes will be driven alternatively into high and low capacitance.

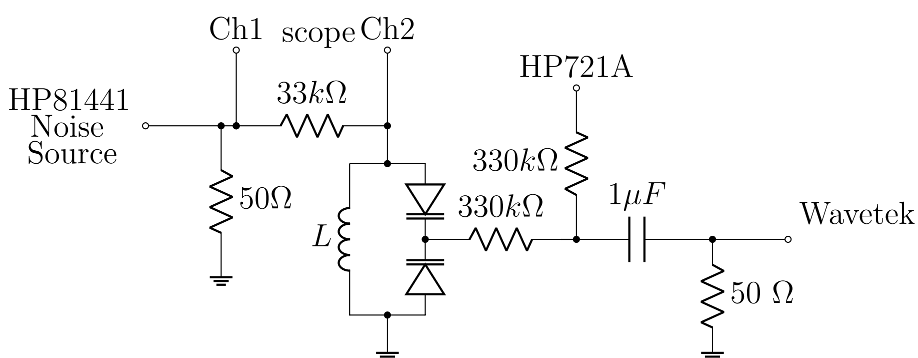

Figure 3.6: Experimental setup of noisy power pump with parametric amplification LC resonator.

Figure 3.6 shows the experimental setup for parametrically pumping the energy flow in and out of the resonator LC driven by the noise source and voltage biased. The varactor diodes are in back-to-back position and are driven by biased DC voltage source "HP721A" and AC voltage source "Wavetek" for the sake of modulation of capacitance via time. $V_1(t)$ from the channel 1 and $V_2(t)$ from channel 2 can be captured by the oscilloscope, from which the instantaneous power pumped to or from LC circuit can be obtained from

$$W(t) = \frac{(V_1 - V_2)}{R_{tr}} V_2 \tag{3.24}$$

Where $R_{tr} = 33k\Omega$. The closest formula for modulated capacitance to the experimental varactor diodes is

$$C(t) = C_0 e^{\epsilon \sin(\mu\omega_0 t)} \tag{3.25}$$

Where $\omega_0 = 1/\sqrt{LC}$, ϵ is related to the capacitance deviation, and $\mu\omega_0 = \omega_d$ is the modulation frequency. The experimental setup is shown in Figure 3.7.

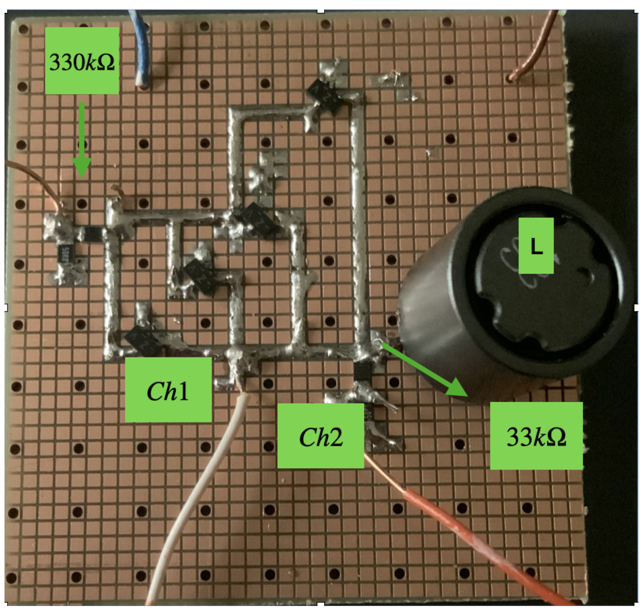

Figure 3.7: The experimental prototype of the circuit in Figure 3.6 with four parallel varactor diodes.

3.6 Parametric Power Pump with a Noise Source Prototype

For the prototype, in order to get nearly 200 pF capacitance, four high quality factor varactor diodes connected parallel to each other but each varactor diode consists of two series or "back-to-back" diodes with $C \approx 100$ pF each so we have nearly 50 pF for each varactor and $50 \times 4 = 200$ pF for the whole system. Note that the bias connection point experiences all eight in parallel at the modulation frequency. The inductor which is used here is $10mH$ and all other details are the same as Figure 3.8.

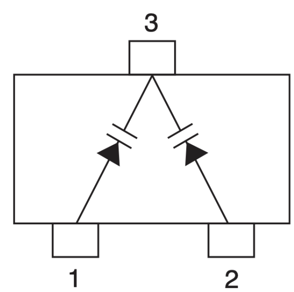

Figure 3.8: Varactor diode, SVC236.

As mentioned before we used oscilloscope to capture the voltage from Ch1 and Ch2. Before starting the experiment it was important to check the varactor diodes if the setup matches the specified capacitance shift. For this process, the voltage from Ch2 with respect to different sinusoidal drive frequencies of the HP81441 is obtained and compared to the data from the C-V plot in the specification sheet of varactor diode. As a result the significant overlap was observed which shows that the varactor diode works well in accordance to the expected specifications. In order to get frequency versus the bias voltage we got data from C-V plot of specifications and fit it with

$$C(V) = C_p + \frac{C_0}{1 + (V/V_0)a} \tag{3.26}$$

Where C_p, C_0, V_0 and a will be obtained via fitting the data representing the parallel capacitance, capacitance change from zero to large bias, characteristic voltage bias, and power law exponent respectively. As a result, the significant overlap was observed which shows that the varactor diode works well in accordance to the expected specifications. It also allowed for the quality factor of the varactor-based resonator to be checked.

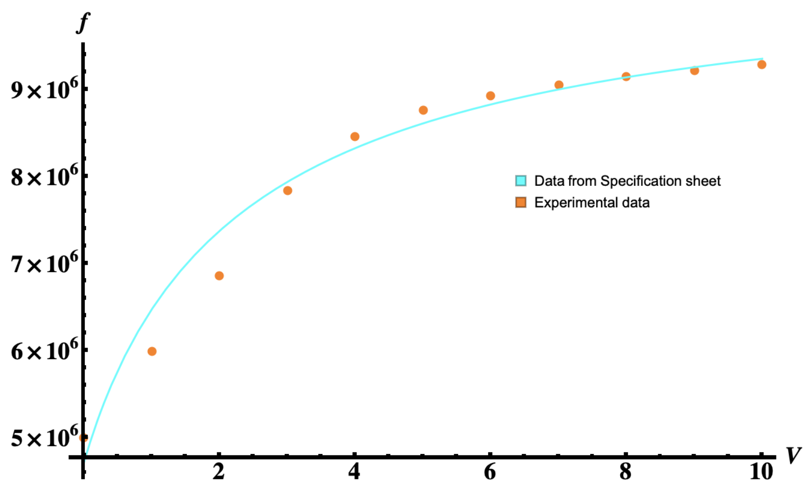

Figure 3.9: Frequency versus the bias voltage from experimental data and fitted data from specification sheet.

In conclusion it is evident the varactor diodes work very well. More study is needed to be done for calculating the exact power pumped in Carnot engine defined earlier. The circuit in Figure 3.7 is ready to be used to test the isothermal and adiabatic paths of the Carnot cycle, but the switching illustrated in Figure 3.4 (right) remains to be developed. This can also be done with varactor diodes, but not perfectly since their capacitance cannot go zero (open switch) to ∞ (closed switch).

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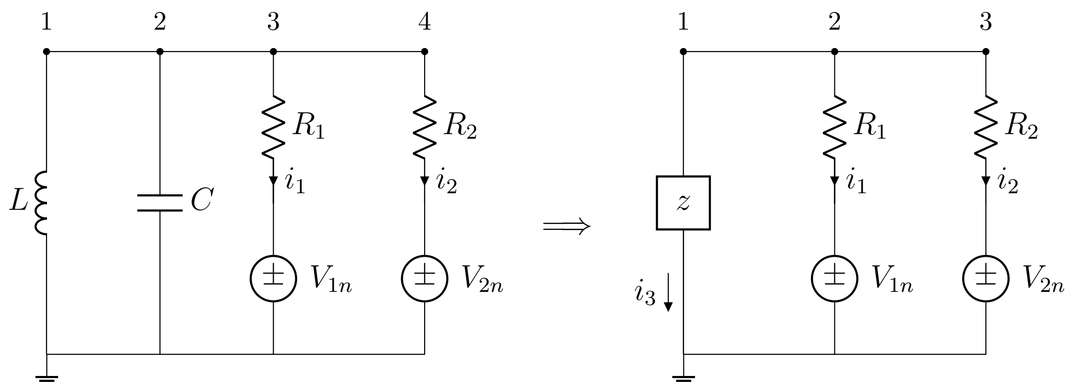
Appendices

Appendix

A Appendix: RLC without External Voltage

Assume we have RLC circuit with two resistors R_1 and R_2 parallel to each other with two noise sources V_{1n} and V_{2n} respectively. Here, from Kirchhoff's law we have:



$$i_1 + i_2 + i_3 = 0 \tag{A.1}$$

$$V - z i_3 = 0 \tag{A.2}$$

$$V - R_1 i_1 - V_{1n} = 0 \tag{A.3}$$

$$V - R_2 i_2 - V_{2n} = 0 \tag{A.4}$$

For calculating V, first we should use eq. (A.1) and solve for i_3 and put it in eq. (A.2). Then solve for i_1. As follows:

$$i_3 = -(i_1 + i_2) \tag{A.5}$$

$$V - zi_3 = 0 \xrightarrow{(5)} V = -z(i_1 + i_2) \tag{A.6}$$

$$i_1 = \frac{V + zi_2}{-z} \tag{A.7}$$

Then put eq. (A.7) in eq. (A.3) and solve for i_2

$$V - R_1(\frac{V + zi_2}{-z}) - V_{1n} = 0 \xrightarrow{V} (z + R_1) + zR_1 i_2 - zV_{1n} = 0 \tag{A.8}$$

$$i_2 = \frac{-V(z + R_1) + zV_{1n}}{R_1 z} \tag{A.9}$$

Then put eq. (A.9) in eq. (A.4) :

$$V - R_2(\frac{-V(z + R_1) + zV_{1n}}{R_1 z}) - V_{2n} = 0$$

$$VR_1 z + V(z + R_1)R_2 - V_{1n}zR_2 - V_{2n}zR1 = 0$$

$$V(R_1 z + R_1 R_2 + zR_2) = (V_{1n}R_2 + V_{2n}R_1)z$$

Then we have:

$$V = \frac{(V_{1n}R_2 + V_{2n}R_1)z}{(R_1 + R_2)z + R_1 R_2} \tag{A.10}$$

We can make it more simplify by making two resistors as parallel ones

$$V = \frac{V_{1n}(\frac{R_2}{R_1+R_2}) + V_{2n}(\frac{R_1}{R_1+R_2})}{(1 + (\frac{R_1 R_2}{R_1+R_2})(\frac{1}{z}))} \tag{A.11}$$

Now we know that z is just the parallel combination of inductance and capacitance impedences, so we have:

$$z^{-1} = i\omega C + \frac{1}{i\omega L} \tag{A.12}$$

$$z^{-1} = i(\omega C - \frac{1}{\omega L}) \tag{A.13}$$

Now put it in eq. (A.11) then we have:

$$V = \frac{V_{1n}(\frac{R_2}{R_1+R_2}) + V_{2n}(\frac{R_1}{R_1+R_2})}{(1 + i(\frac{R_1R_2}{R_1+R_2})(\omega C - \frac{1}{\omega L}))} \tag{A.14}$$

Now it is the time to calculate VV^*

$$\left((\frac{V_{1n}(\frac{R_2}{R_1+R_2}) + V_{2n}(\frac{R_1}{R_1+R_2})}{(1 + i(\frac{R_1R_2}{R_1+R_2})(\omega C - \frac{1}{\omega L}))}) \right)^* \left(\frac{V_{1n}(\frac{R_2}{R_1+R_2}) + V_{2n}(\frac{R_1}{R_1+R_2})}{(1 - i(\frac{R_1R_2}{R_1+R_2})(\omega C - \frac{1}{\omega L}))} \right)$$

Then we have

$$\left(\frac{\left(\frac{|R_2|^2}{|R_1+R_2|^2}\right) V_{1n}^2 + \left(\frac{|R_1|^2}{|R_1+R_2|^2}\right) V_{2n}^2 + 2\frac{|R_1R_2|}{|R_1+R_2|^2} V_{1n}V_{2n}}{1 + \frac{|R_1R_2|^2}{|R_1+R_2|^2}\left(\omega C - \frac{1}{\omega L}\right)^2} \right)$$

Then we take a mean of above equation which make the third term in numerator zero as there is no correlation between V_{1n} and V_{2n}, then we have:

$$V = \left(\frac{1}{1 + \frac{|R_1R_2|^2}{|R_1+R_2|^2}\left(\omega C - \frac{1}{\omega L}\right)^2} \right) \left(\frac{|R_2|^2}{|R_1+R_2|^2} \right) \langle V_{1n}^2 \rangle$$
$$+ \left(\frac{1}{1 + \frac{|R_1R_2|^2}{|R_1+R_2|^2}\left(\omega C - \frac{1}{\omega L}\right)^2} \right) \left(\frac{|R_1|^2}{|R_1+R_2|^2} \right) \langle V_{2n}^2 \rangle$$

We know that $S_v(0) = 4k_BRT$ and $\langle V_n^2 \rangle = \frac{1}{2\pi}\int_0^\infty S_v(0)d\omega$ so for the first term above we have :

$$\left(\frac{|R_1|^2}{|R_1+R_2|^2}\right)\left(\frac{4k_BT|R_1|}{2\pi}\right)\int_0^\infty \frac{\omega^2L^2d\omega}{\omega^2L^2+\frac{|R_1R_2|^2}{|R_1+R_2|^2}\left(\omega^2LC-1\right)^2} \tag{A.15}$$

Then we define u^2 as ω^2LC we have, $2LC\omega d\omega = 2udu$ and $\omega = \sqrt{\frac{1}{LC}}u$. Then extract $\dfrac{|R_1R_2|^2}{|R_1+R_2|^2}$ and call $\gamma^2 = \dfrac{L/C}{\dfrac{|R_1R_2|^2}{|R_1+R_2|^2}}$. Then the integrate have this form

$$\int_0^\infty \frac{u^2du}{u^2\gamma^2+(u^2-1)^2} = \frac{\pi}{2\gamma} \tag{A.16}$$

So the first term of eq. (A.15) can be written as :

$$\left(\frac{|R_1|^2}{\cancel{|R_1+R_2|^2}}\right)\left(\frac{\cancel{4}k_BT|R_1|}{\cancel{2\pi}}\right)\cancel{L^2}\left(\frac{\cancel{|R_1+R_2|^2}}{|R_1R_2|^2}\right)\frac{1}{\cancel{LC}}\sqrt{\frac{1}{LC}}\;\; \frac{1}{C}\sqrt{\frac{L}{C}} \;\; \left(\frac{\cancel{\pi}}{\cancel{2}\gamma}\right)$$

we have the same argument for the second term ofeq. (A.15). We have:

$$\left(\frac{|R_2|^2}{|R_1R_2|^2}|R_1|\right)\left(\frac{k_BT}{C}\right)\frac{1}{\gamma}\sqrt{\frac{L}{C}}+\left(\frac{|R_1|^2}{|R_1R_2|^2}|R_2|\right)\left(\frac{k_BT}{C}\right)\frac{1}{\gamma}\sqrt{\frac{L}{C}}$$

By a little bit manipulation we have :

$$\left(\frac{|R_2|^2}{|R_1R_2|^2}|R_1|\right)\left(\frac{k_BT}{C}\right)\frac{1}{\gamma}\left(\frac{|R_1|}{|R_1|}\right)\sqrt{\frac{L}{C}} + \left(\frac{|R_1|^2}{|R_1R_2|^2}|R_2|\right)\left(\frac{k_BT}{C}\right)\frac{1}{\gamma}\left(\frac{|R_2|}{|R_2|}\right)\sqrt{\frac{L}{C}}$$

Then we call $\gamma_1 = \frac{1}{|R_1|}\sqrt{\frac{L}{C}}$ and $\gamma_2 = \frac{1}{|R_2|}\sqrt{\frac{L}{C}}$. Then after these dimensionless quantities we have,

$$\langle V^2\rangle = \left(\frac{k_BT}{C}\right)\left(\frac{|R_2|^2|R_1|^2}{|R_1R_2|^2}\left(\frac{\gamma_1}{\gamma}\right)+\frac{|R_1|^2|R_2|^2}{|R_1R_2|^2}\left(\frac{\gamma_2}{\gamma}\right)\right) \tag{A.17}$$

So we have :

$$\langle V^2 \rangle = \left(\frac{k_B T}{C} \right) \left(\frac{\gamma_1 + \gamma_2}{\gamma} \right) \tag{A.18}$$

B Appendix: RLC with External Voltage

Consider a non-$\mathscr{PT}$-symmetric RLC circuit with gain and loss with an external signal $V_s e^{i\omega_s t}$. The signal source inherently adds signal noise V_{sn} and external resistor r to the system.

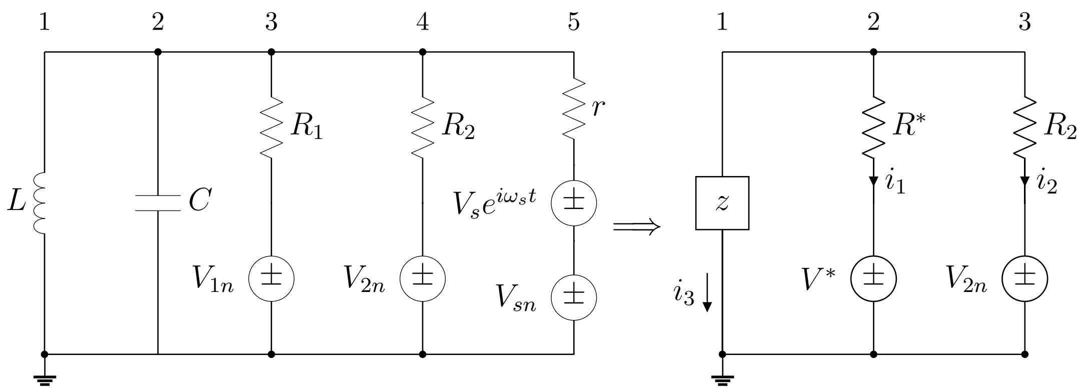


Figure B.1: Non-$\mathscr{PT}$-symmetric RLC circuit with a voltage source. Gain and loss is depicted as R_2 and R_1 parallel with r in the right picture which is R^* in the right picture. z here is the LC oscillator for simplicity

Here R^* is the parallel combination of R_1 and r which is $R^* = \left(\frac{R_1 * r}{R_1 + r}\right)$, and V^* is calculated in this way, first put $V' = V_s e^{i\omega_s t} + V_{sn}$ for convenience, then we have:

$$\begin{aligned} &i_1 + i_2 = 0 \rightarrow i_1 = -i_2 \\ &V - i_2 r - V' = 0 \end{aligned} \tag{B.1}$$

By combining above equations we have

$$\begin{aligned} &V + \left(\frac{V - V'}{r}\right) R_1 - V_{1n} r = 0 \rightarrow Vr + VR_1 - V'R_1 - V_{1n} r = 0 \\ &V = V' \left(\frac{R_1'}{R_1 + r}\right) + V_{1n} \left(\frac{r}{r + R_1}\right) \end{aligned} \tag{B.2}$$

V^* is the summation of

$$V^* = \left(V_s e^{i\omega_s t} + V_{sn}\right)\left(\frac{R_1}{R_1 + r}\right) + V_{1n}\left(\frac{r}{R_1 + r}\right)$$

The trick here is making the present system resemble to the previous problem in order to make it easier. Therefore, substitute R^* and V^* with R_1 and V_2 respectively in eq. (A.14). Then we have:

$$V = \frac{V^*(\dfrac{R_2}{R^* + R_2}) + V_{2n}(\dfrac{R^*}{R^* + R_2})}{1 + i(\dfrac{R^* R_2}{R^* + R_2})(\omega C - \dfrac{1}{\omega L})} \tag{B.3}$$

Put V^* and R^* in above equation then we have

$$V = \left(\frac{\left((V_s e^{i\omega_s t} + V_{sn})\left(\frac{R_1}{R_1+r}\right) + V_{1n}\left(\frac{r}{r+R_1}\right)\right)\left(\frac{R_2}{\left(\frac{R_1 * r}{R_1+r}\right)+R_2}\right) + V_{2n}\left(\frac{\left(\frac{R_1 * r}{R_1+r}\right)}{\left(\frac{R_1 * r}{R_1+r}\right)+R_2}\right)}{\left(1 + i\left(\frac{\left(\frac{R_1 * r}{R_1+r}\right)R_2}{\left(\frac{R_1 * r}{R_1+r}\right)+R_2}\right)(\omega C - \frac{1}{\omega L})\right)}\right)$$

By simplifying the above equation we have:

$$V = \frac{\left(V_s e^{i\omega_s t}\left(\frac{R_1}{R_1+r}\right) + V_{sn}\left(\frac{R_1}{R_1+r}\right) + V_{1n}\left(\frac{r}{r+R_1}\right)\right)\left(\frac{R_2(r+R_1)}{r(R_1+R_2)+R_1R_2}\right)}{1 + i(\frac{R_1R_2r}{r(R_1+R_2)+R_1R_2})(\omega C - \frac{1}{\omega L})} + \frac{V_{2n}\left(\frac{R_1 r}{r(R_1+R_2)+R_1R_2}\right)}{1 + i(\frac{R_1R_2r}{r(R_1+R_2)+R_1R_2})(\omega C - \frac{1}{\omega L})}$$

More simplification gets:

$$V = \frac{V_{2n}(\frac{R_1 r}{r(R_1+R_2)+R_1R_2})}{(1 + i(\frac{R_1R_2r}{r(R_1+R_2)+R_1R_2})(\omega C - \frac{1}{\omega L}))} + \frac{(\frac{R_2(r+R_1)}{r(R_1+R_2)+R_1R_2})}{(1 + i(\frac{R_1R_2r}{r(R_1+R_2)+R_1R_2})(\omega C - \frac{1}{\omega L}))} \times$$
$$V_s e^{i\omega_s t}\left(\frac{R_1R_2}{r(R_1 + R_2) + R_1R_2}\right) + V_{sn}\left(\frac{R_1R_2}{r(R_1 + R_2) + R_1R_2}\right)$$
$$+ V_{1n}\left(\frac{rR_2}{r(R_1 + R_2) + R_1R_2}\right) \tag{B.4}$$

Now it is the time to calculate VV^*

$$\langle V^2 \rangle = \tag{B.5}$$
$$\left(\frac{1}{1 + \left(\frac{|rR_1R_2|^2}{|r(R_1+R_2)+R_1R_2|^2} \right) \left(\omega C - \frac{1}{\omega L} \right)^2} \right) \left(\frac{|R_1R_2|^2}{|r(R_1+R_2)+R_1R_2|^2} \right) \langle V_s^2 \rangle +$$
$$\left(\frac{1}{1 + \left(\frac{|rR_1R_2|^2}{|r(R_1+R_2)+R_1R_2|^2} \right) \left(\omega C - \frac{1}{\omega L} \right)^2} \right) \left(\frac{|R_1R_2|^2}{|r(R_1+R_2)+R_1R_2|^2} \right) \langle V_{sn}^2 \rangle +$$
$$\left(\frac{1}{1 + \left(\frac{|rR_1R_2|^2}{|r(R_1+R_2)+R_1R_2|^2} \right) \left(\omega C - \frac{1}{\omega L} \right)^2} \right) \left(\frac{|rR_2|^2}{|r(R_1+R_2)+R_1R_2|^2} \right) \langle V_{1n}^2 \rangle +$$
$$\left(\frac{1}{1 + \left(\frac{|rR_1R_2|^2}{|r(R_1+R_2)+R_1R_2|^2} \right) \left(\omega C - \frac{1}{\omega L} \right)^2} \right) \left(\frac{|R_1r|^2}{|r(R_1+R_2)+R_1R_2|^2} \right) \langle V_{2n}^2 \rangle$$

Just we should repeat the similar work for noise voltages, but here the temperature for signal (T_s) is different from the internal temperature (T_i). Moreover, $\langle V_{sn}^2 \rangle$ is also constant and does not have any dependency on temperature and we have:

$$\langle V_{out}^2 \rangle = \tag{B.6}$$
$$\left(\frac{k_B T_i}{C} \right) \left(\frac{\gamma_1 + \gamma_2}{\gamma} \right) + \left(\frac{k_B T_s}{C} \right) \left(\frac{\gamma_r}{\gamma} \right) + \left(\frac{\langle V_s^2 \rangle \frac{|R_1R_2|^2}{|r(R_1+R_2)+R_1R_2|^2}}{1 + (\omega_s C - \frac{1}{\omega_s L})^2 (\frac{|rR_1R_2|^2}{|r(R_1+R_2)+R_1R_2|^2})} \right)$$

Where $\gamma_1 = \frac{1}{|R_1|}\sqrt{\frac{L}{C}}$, $\gamma_2 = \frac{1}{|R_2|}\sqrt{\frac{L}{C}}$,$\gamma_r = \frac{1}{|r|}\sqrt{\frac{L}{C}}$ and $\gamma = \frac{1}{\frac{|rR_1R_2|}{|r(R_1+R_2)+R_1R_2|}}\sqrt{\frac{L}{C}}$.

Here in this equation for the last part put $S_v(\omega) = \langle V_s^2 \rangle \delta(\omega - \omega_s)$ and after taking integration we the ω in the denominator is substituted with ω_s.

C Appendix: $\mathscr{PT}$-symmetric RLC Circuit with one external source

After discussing the non-$\mathscr{PT}$-symmetric RLC oscillator in the last section, it is time to study $\mathscr{PT}$-symmetric RLC with only one source in the loss side. This is the first step to determine SNR in $\mathscr{PT}$-symmetric RLC circuits with two signal sources.

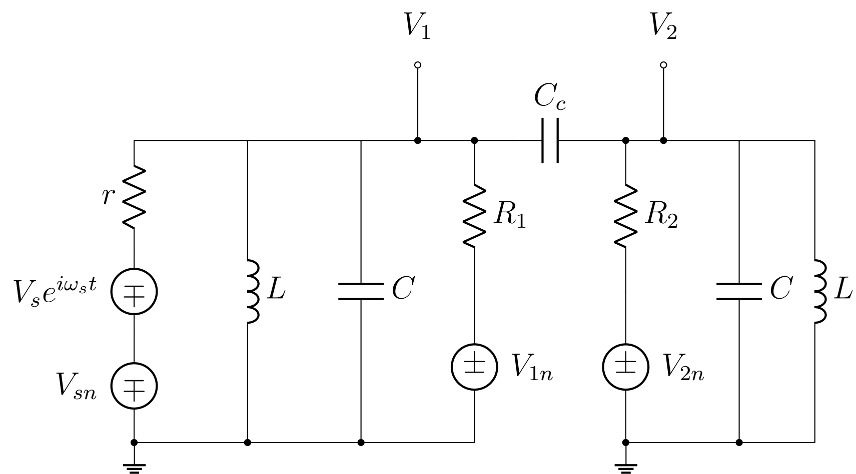


Figure C.1: $\mathscr{PT}$-symmetric RLC Circuit with one external source

$$i\omega C_c(V_1 - V_2) + i\omega C V_1 + \frac{V_1}{i\omega L} + \frac{(V_1 - V_{sn} - V_s e^{i\omega_s t})}{r} + \frac{(V_1 - V_{1n})}{R_1} = 0$$
$$i\omega C_c(V_2 - V_1) + i\omega C V_2 + \frac{V_2}{i\omega L} + \frac{(V_2 - V_{2n})}{R_2} = 0$$

We consider $\Omega = \frac{\omega}{\omega_0}$ where $\omega_0 = \frac{1}{\sqrt{LC}}$ and $c = \frac{C_c}{C}$.

$$\begin{pmatrix} -\Omega(c+1) + \frac{1}{\Omega} + i\gamma' & \Omega c \\ \Omega c & -\Omega(c+1) + \frac{1}{\Omega} + i\gamma_2 \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} =$$

$$i \begin{pmatrix} V_{sn}\gamma_r + V_s e^{i\omega_s t}\gamma_r + V_{1n}\gamma_1 \\ V_{2n}\gamma_2 \end{pmatrix}$$

Now it is the time to take a reverse of it:

$$\begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \frac{i}{(1+2c)^2(\Omega-\Omega_1)(\Omega-\Omega_2)(\Omega-\Omega_3)(\Omega-\Omega_4)} \times$$
$$\begin{pmatrix} -\Omega(c+1)+\frac{1}{\Omega}+i\gamma_2 & -\Omega c \\ -\Omega c & -\Omega(c+1)+\frac{1}{\Omega}+i\gamma^{'} \end{pmatrix} \begin{pmatrix} V_{sn}\gamma_r + V_s e^{i\omega_s t}\gamma_r + V_{1n}\gamma_1 \\ V_{2n}\gamma_2 \end{pmatrix}$$

Thus it is easy to find $\langle V_1^2 \rangle$ and $\langle V_2^2 \rangle$ by solving the above matrix. Moreover, $\gamma^{'} = -\gamma_2$. If we call $(\Omega-\Omega_1)(\Omega-\Omega_2)(\Omega-\Omega_3)(\Omega-\Omega_4) = \tilde{\Omega}$ as follows:

$$\langle V_1^2 \rangle =$$
$$\frac{1}{(1+2c)^2|\tilde{\Omega}|^2}\left((-\Omega(c+1)+\frac{1}{\Omega})^2+\gamma_2^2\right)\left(\langle V_{sn}^2\rangle\gamma_r^2 + \frac{1}{2}\langle V_s^2\rangle\delta(\omega-\omega_0)\gamma_r^2 + \langle V_{1n}^2\rangle\gamma_1^2\right)$$
$$+\Omega^2\gamma_2^2c^2\langle V_{2n}^2\rangle$$

we have :

$$\langle V_{sn}^2 \rangle = \frac{1}{2\pi}\int 4k_B T_s r d\omega \tag{C.1}$$
$$\langle V_{1n}^2 \rangle = \frac{1}{2\pi}\int 4k_B T_s R_1 d\omega \tag{C.2}$$
$$\langle V_{2n}^2 \rangle = \frac{1}{2\pi}\int 4k_B T_s R_2 d\omega \tag{C.3}$$

So we have the first term:

$$\frac{4k_B T_s |r|\omega_0\gamma_r^2}{2\pi(1+2c)^2}\int \frac{\left((-\Omega(c+1)+\frac{1}{\Omega})^2+\gamma_2^2\right)}{|\tilde{\Omega}|^2} d\Omega \tag{C.4}$$

Second Equation:

$$\frac{V_s^2\gamma_r^2\omega_0}{2(1+2c)^2}\int \frac{\left((-\Omega(c+1)+\frac{1}{\Omega})^2+\gamma_2^2\right)\delta(\omega-\omega_0)}{|\tilde{\Omega}|^2}\,d\Omega \tag{C.5}$$

Third one:

$$\frac{4k_BT|R_1|\omega_0\gamma_1^2}{2\pi(1+2c)^2}\int \frac{\left((-\Omega(c+1)+\frac{1}{\Omega})^2+\gamma_2^2\right)}{|\tilde{\Omega}|^2}\,d\Omega \tag{C.6}$$

Fourth one:

$$\frac{4k_BT|R_2|\omega_0c^2\gamma_2^2}{2\pi(1+2c)^2}\int \frac{\Omega^2}{|\tilde{\Omega}|^2}\,d\Omega \tag{C.7}$$

Here it is assumed that the non-Hermitian system is not balanced so that $\gamma_{loss} \neq -\gamma_{gain}$ so we define Δ as an infinitesimal deviation from balanced γ. Also, call $\gamma_2 = \gamma$ So, we have

$$\gamma_{loss} = -\gamma + \Delta, \qquad \gamma_{gain} = \gamma + \Delta \tag{C.8}$$

So our Hamiltonian is as follows:

$$\begin{pmatrix} -\Omega(c+1)+\frac{1}{\Omega}-i\gamma+i\Delta & \Omega c \\ \Omega c & -\Omega(c+1)+\frac{1}{\Omega}+i\gamma+i\Delta \end{pmatrix}\begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = i\begin{pmatrix} V_{sn}\gamma_r + V_s e^{i\omega_s t}\gamma_r + V_{1n}\gamma_1 \\ V_{2n}\gamma \end{pmatrix} \tag{C.9}$$

Adding Δ will break the symmetry of our non-Hermitian system. Now the task is to compute V_1 and V_2. For getting the result we have to first come up with the idea that how get the roots of the first matrix, as the roots will not be as straightforward as it was for the symmetric case. First compute the determinant of the matrix which is polynomial for when $s = i\Omega$

$$(1+2c)s^4 + 2(1+c)\Delta s^3 + (\Delta^2 - \gamma^2 + 2(1+c))s^2 + 2\Delta s + 1 = 0 \quad \text{(C.10)}$$

Then we can make it easier by doing change of variable by $u = (1+2c)^{\frac{1}{4}}$, then we have:

$$u^4 s^4 + (1+u^4)\Delta s^3 + (\Delta^2 - \gamma^2 + (1+u^4))s^2 + 2\Delta s + 1 = 0 \quad \text{(C.11)}$$

So here I want to expand Ω in terms of Δ, we have $\Omega(\Delta) = \Omega_\pm + \Delta.\lambda$ where $\Omega_\pm = \frac{\sqrt{\gamma_c^2-\gamma^2} \pm \sqrt{\gamma_{pt}^2-\gamma^2}}{2\sqrt{1+2c}}$, $\gamma_c = \sqrt{1+2c}+1$ and $\gamma_{pt} = \sqrt{1+2c}-1$. So the determinant of the unbalanced matrix (call it M) would be $detM = detM_0 + \Delta.\lambda + \mathcal{O}(\Delta^2)$. I got an analytic result for $\lambda_\pm$ which is:

$$\lambda_\pm(\gamma) = \frac{i(\Omega_\pm^2(1+c)-1)}{2\Omega_\pm^2(1+2c) - 2(1+c) + \gamma^2}$$

Now its time to define $\Omega(\gamma,\Delta) = \Omega_\pm(\gamma) + \Delta * \lambda_\pm(\gamma,\Delta)$ and we have:

$$\Omega(\gamma,\Delta) = \Omega_\pm + \left(\frac{i(\Omega_\pm^2(1+c)-1)}{2\Omega_\pm^2(1+2c) - 2(1+c) + \gamma^2}\right) * \Delta \quad \text{(C.12)}$$

Also the other two poles in balanced system is $\Omega^\pm = -\frac{\sqrt{\gamma_c^2-\gamma^2} \pm \sqrt{\gamma_{pt}^2-\gamma^2}}{2\sqrt{1+2c}}$ and $\lambda^\pm(\gamma) = \frac{i((\Omega^\pm)^2(1+c)-1)}{2(\Omega^\pm)^2(1+2c)-2(1+c)+\gamma^2}$ the same as the other two poles. So,

$$\Omega(\gamma,\Delta) = \Omega^\pm + \left(\frac{i((\Omega^\pm)^2(1+c)-1)}{2(\Omega^\pm)^2(1+2c) - 2(1+c) + \gamma^2}\right) * \Delta$$

So we have our four poles as follows:

$$\Omega_{1,2,3,4}(\gamma,\Delta) = \Omega_\pm^\pm + \left(\frac{i((\Omega_\pm^\pm)^2(1+c)-1)}{2(\Omega_\pm^\pm)^2(1+2c) - 2(1+c) + \gamma^2}\right) * \Delta \quad \text{(C.13)}$$

Now there exists four poles which are $\Omega_\pm^\pm$, we have to compute eq. (C.4) through eq. (C.7) with these unbalanced roots rather than the balanced ones. First start computing the integrals by starting with the easiest one, eq. (C.7), but before starting it is better to generalize the integration. Call Ω-dependence function $f(\Omega)$ then the integration is :

$$\frac{1}{(1+2c)^2}\oint \frac{f(\Omega)d\Omega}{(\Omega-\Omega_1)(\Omega-\Omega_1^*)(\Omega-\Omega_2)(\Omega-\Omega_2^*)(\Omega-\Omega_3)(\Omega-\Omega_3^*)(\Omega-\Omega_4)(\Omega-\Omega_4^*)}$$

if the pole which we want to compute is Ω_i where i can be 1 to 4, then we have:

$$\begin{aligned}\oint \frac{f(\Omega)d\Omega}{(\Omega-\Omega_i)(\Omega-\Omega_i^*)} &= (2\pi i)\left(\frac{f(\Omega)d\Omega}{(\Omega-\Omega_i)(\Omega-\Omega_i^*)}(\Omega-\Omega_i)\right)\Bigg|_{\Omega=\Omega_i} \\ &= \cancel{2}\pi\cancel{i}\frac{f(\Omega_i)}{\cancel{2i}\,\mathrm{Im}(\Omega_i)} = \frac{\pi f(\Omega_i)}{\mathrm{Im}(\Omega_i)} \end{aligned} \tag{C.14}$$

Where $\Omega-\Omega_i^* = 2i\,\mathrm{Im}(\Omega i)$. When we are computing the integration at a pole then we can approximate the the other pole exactly to be real so we the integration is:

$$\frac{1}{(1+2c)^2}\sum_{i=1}^{4}\pi\frac{f(\Omega_i)}{\mathrm{Im}(\Omega_i)}\prod_{\substack{j=1\\ j\neq i}}^{4}\frac{1}{(\Omega_i-\Omega_j)^2} \tag{C.15}$$

D Appendix: Analytic Calculation for Signal to Noise Ratio

For calculating signal to noise ratio (SNR), we have to first compute the integral for the noise (gain) part and signal part. First I want to use eq. (C.15) and simplify the integral, then computing the signal part which is a delta function.

D.1 Calculating Noise in Gain

For computing the noise via eq. (C.15) we know that $f(\Omega) = \Omega^2$ and I want to get an expression for $P = \frac{1}{(1+2c)^2}\prod_{\substack{j=1\\ j\neq i}}^{4}\frac{1}{(\Omega_i-\Omega_j)^2}$. This expression falls into two categories with just two frequencies Ω_+ and Ω_- as follows:

For both "Ω_+" and "Ω^+" $\implies$ $P = \frac{1}{(1+2c)^2}\frac{1}{(2X)^2(2Y)^2(2\Omega_+)^2}$

For both "Ω_-" and "Ω^-" $\implies$ $P = \frac{1}{(1+2c)^2}\frac{1}{(2X)^2(2Y)^2(2\Omega_-)^2}$

Where $X = \frac{\sqrt{\gamma_c^2-\gamma^2}}{2\sqrt{1+2c}}$ and $Y = \frac{\sqrt{\gamma_{pt}^2-\gamma^2}}{2\sqrt{1+2c}}$ so we have:

$$\frac{1}{(2X)^2(2Y)^2} = \frac{(2c+1)^2}{\gamma^4 - 4(c+1)\gamma^2 + 4c^2} \tag{D.1}$$

So for the P for both cases we have:

$$P(\Omega_\pm) = \frac{1}{(\gamma^4 - 4(c+1)\gamma^2 + 4c^2)\,(2\Omega_\pm)^2} \tag{D.2}$$

Also we know that the $\mathrm{Im}(\Omega_\pm)$ is nothing but the $\left(\frac{(\Omega_\pm^2(1+c)-1)}{2\Omega_\pm^2(1+2c)-2(1+c)+\gamma^2}\right) * \Delta$. Also we should not forget that we have other terms for Ω^- and Ω^+ as we have to multiply the equation to the factor 2 then put everything in eq. (C.15), so we will get:

$$\frac{2\pi}{4(\gamma^4 - 4(c+1)\gamma^2 + 4c^2)\Delta} \sum_{i=\pm} \frac{(2\Omega_\pm^2(1+2c) - 2(1+c) + \gamma^2)}{(\Omega_\pm^2(1+c) - 1)} \tag{D.3}$$

I put this expression in Mathematica and simplify it to :

$$\frac{2(4c^3 + c\gamma^2(-8+\gamma^2) + \gamma^2(-4+\ \gamma^2) - 4c^2(-1+\gamma^2))}{(c^2 - \gamma^2(1+c))}$$

Then I put this integral into (1-38). We have:

$$\begin{aligned} &\frac{4k_B T|R_2|\omega_0 c^2\gamma_2^2}{2\pi} \int \frac{\Omega^2}{|\tilde{\Omega}|^2}\, d\Omega = \\ &(\frac{k_B T\gamma_2}{C})(\frac{c^2}{\Delta})(\frac{1}{(\gamma^4 - 4(c+1)\gamma^2 + 4c^2)}) \\ &\times (\frac{2(4c^3 + c\gamma^2(-8+\gamma^2) + \gamma^2(-4+\ \gamma^2) - 4c^2(-1+\gamma^2))}{(c^2 - \gamma^2(1+c))}) \end{aligned} \tag{D.4}$$

D.2 Calculating The Internal Noise In the Loss Resistor

$$\frac{4k_BT|R_1|\omega_0 c^2\gamma_1^2}{2\pi}\int \frac{\Omega^2}{|\tilde{\Omega}|^2}\,d\Omega =$$
$$(\frac{k_BT\gamma_1}{C})(\frac{c^2}{\Delta})(\frac{1}{(\gamma^4-4(c+1)\gamma^2+4c^2)})$$
$$\times\,(\frac{2(4c^3+c\gamma^2(-8+\gamma^2)+\gamma^2(-4+\ \gamma^2)-4c^2(-1+\gamma^2))}{(c^2-\gamma^2(1+c))}) \tag{D.5}$$

D.3 Calculating The Internal Noise In the External Resistor

$$\frac{4k_BT_s|r|\omega_0 c^2\gamma_r^2}{2\pi}\int \frac{\Omega^2}{|\tilde{\Omega}|^2}\,d\Omega =$$
$$(\frac{k_BT_s\gamma_r}{C})(\frac{c^2}{\Delta})(\frac{1}{(\gamma^4-4(c+1)\gamma^2+4c^2)})$$
$$\times\,(\frac{2(4c^3+c\gamma^2(-8+\gamma^2)+\gamma^2(-4+\ \gamma^2)-4c^2(-1+\gamma^2))}{(c^2-\gamma^2(1+c))}) \tag{D.6}$$

D.4 Calculating Signal RMS Voltage

For calculating eq. (C.6) we have to first we have to change the variable of the $\delta(\omega-\omega_\pm)$ to $\frac{1}{\omega_0}\delta(\Omega-\Omega_\pm)$. Moreover, we have $\Omega_i-\Omega_i$ term where i is the pole that we want to measure at, so this term is just $|\Delta|^2$ for all poles and also the integrate falls into two categories like the Noise part. Then we have:

$$\frac{\langle V_s\rangle^2\gamma_r^2}{2}\int \frac{\left((-\Omega(c+1)+\frac{1}{\Omega})^2+\gamma^2\right)\delta(\Omega-\Omega_\pm)}{|\tilde{\Omega}|^2}\,d\Omega = \frac{\langle V_s\rangle^2\gamma_r^2}{2}\times$$
$$\sum_{i=\pm}\frac{2\left((-\Omega_i(c+1)+\frac{1}{\Omega_i})^2+\gamma^2\right)}{|\left(\frac{(\Omega_i^2(1+c)-1)}{2\Omega_i^2(1+2c)-2(1+c)+\gamma^2}\right)*\Delta|^2(2\Omega_i)^2(\gamma^4-4(c+1)\gamma^2+4c^2)} \tag{D.7}$$

After putting this into Mathematica we have:

$$\frac{\langle V_s\rangle^2\gamma_r^2}{2}(\frac{1}{\Delta^2})\frac{1}{4(\gamma^4-4(c+1)\gamma^2+4c^2)}\times$$
$$(\frac{2c^2(4c^3+c\gamma^2(-8+\gamma^2)+\gamma^2(-4+\gamma^2)-4c^2(-1+\gamma^2))}{(c^2-\gamma^2(1+c))}) \tag{D.8}$$

D.5 SNR for $\mathscr{P}\mathscr{T}$-symmetric Coupled Oscillator

$$SNR|_{\text{Coupled Oscillators}} = \frac{\langle V_s\rangle^2\gamma_r^2 C}{k_B T(\gamma_1+\gamma_2+\gamma_r)\Delta} \tag{D.9}$$

$$SNR|_{\text{Single Oscillator}} = \frac{\langle V_s\rangle^2\gamma_r^2 C}{k_B T(\gamma_1+\gamma_2+\gamma_r)\Delta} \tag{D.10}$$

So we have:

$$\text{Noise Coupled OSC} = (\frac{k_B T(\gamma_1+\gamma_2+\gamma_r)}{C})(\frac{c^2}{\Delta})(\frac{1}{4(\gamma^4-4(c+1)\gamma^2+4c^2)})$$
$$\times(\frac{2(4c^3+c\gamma^2(-8+\gamma^2)+\gamma^2(-4+\gamma^2)-4c^2(-1+\gamma^2))}{(c^2-\gamma^2(1+c))}) \tag{D.11}$$

$$\text{Noise One OSC} = \frac{K_B T}{C}\frac{\gamma_2}{\Delta} \tag{D.12}$$

$$\text{Signal Coupled OSC} =$$
$$\langle V_s\rangle^2\gamma_r^2(\frac{1}{\Delta^2})\frac{1}{4(\gamma^4-4(c+1)\gamma^2+4c^2)} \tag{D.13}$$
$$\times(\frac{2c^2(4c^3+c\gamma^2(-8+\gamma^2)+\gamma^2(-4+\gamma^2)-4c^2(-1+\gamma^2))}{(c^2-\gamma^2(1+c))})$$

$$\text{Signal One OSC} = \frac{\langle V_s\rangle^2(\gamma_1+\gamma_2+\gamma_r)}{\Delta^2} \tag{D.14}$$

E Appendix: $\mathscr{P}\mathscr{T}$-symmetric RLC System with two noise sources

Consider a $\mathscr{P}\mathscr{T}$-symmetric RLC circuit with two signal sources which have a phase difference ϕ as shown in Figure E.1

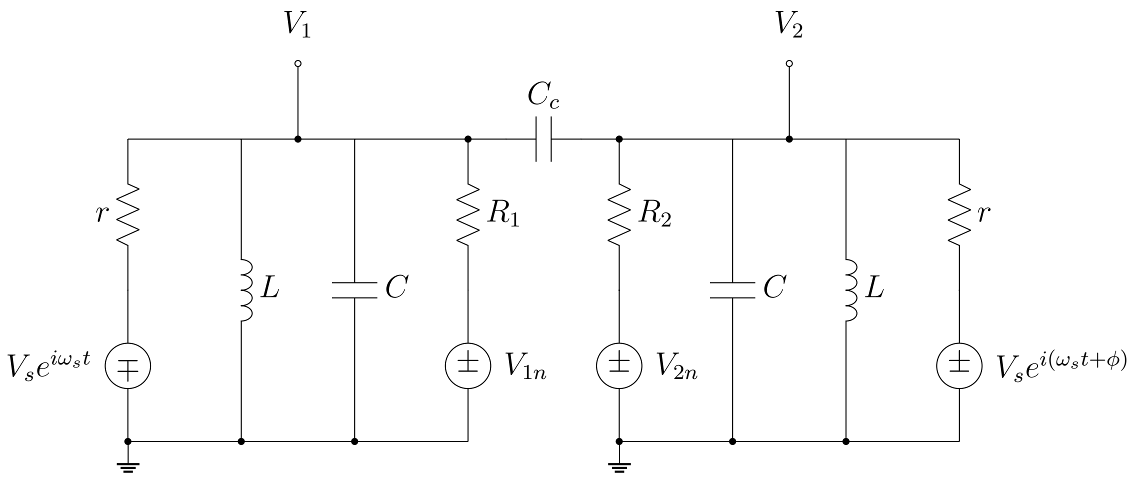


Figure E.1: $\mathscr{PT}$-symmetric RLC circuit with two signal sources with a phase difference ϕ

$$
\begin{aligned}
i\omega C_c(V_1 - V_2) + i\omega C V_1 + \frac{V_1}{i\omega L} + \frac{(V_1 - V_s e^{i\omega_s t})}{r} + \frac{(V_1 - V_{1n})}{-R} = 0 \\
i\omega C_c(V_2 - V_1) + i\omega C V_2 + \frac{V_2}{i\omega L} + \frac{(V_2 - V_s e^{i(\omega_s t+\phi)})}{r} + \frac{(V_2 - V_{2n})}{R} = 0
\end{aligned}
\tag{E.1}
$$

We consider $\Omega = \frac{\omega}{\omega_0}$ where $\omega_0 = \frac{1}{\sqrt{LC}}$ and $c = \frac{C_c}{C}$.

$$
\begin{pmatrix} -\Omega(c+1) + \frac{1}{\Omega} - i\gamma & \Omega c \\ \Omega c & -\Omega(c+1) + \frac{1}{\Omega} + i\gamma \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = i \begin{pmatrix} V_s e^{i\omega_s t}\gamma_r - V_{1n}\gamma \\ V_s e^{i(\omega_s t+\phi)}\gamma_r + V_{2n}\gamma \end{pmatrix}
$$

Now it is the time to take a reverse of it:

$$
\begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \frac{i}{(\Omega-\Omega_1)(\Omega-\Omega_2)(\Omega-\Omega_3)(\Omega-\Omega_4)} \times \tag{E.2}
$$

$$
\begin{pmatrix} -\Omega(c+1) + \frac{1}{\Omega} + i\gamma & \Omega c \\ \Omega c & -\Omega(c+1) + \frac{1}{\Omega} - i\gamma \end{pmatrix} \begin{pmatrix} V_s e^{i\omega_s t}\gamma_r - V_{1n}\gamma \\ V_s e^{i(\omega_s t+\phi)}\gamma_r + V_{2n}\gamma \end{pmatrix}
$$

Thus it is easy to find $\langle V_1^2\rangle$ and $\langle V_2^2\rangle$ by solving the above matrix. If we call $(\Omega-\Omega_1)(\Omega-\Omega_2)(\Omega-\Omega_3)(\Omega-\Omega_4)=\tilde{\Omega}$ as follows:

$$\begin{aligned}
&\langle V_1^2\rangle = \\
&\frac{1}{(1+2c)^2|\tilde{\Omega}|^2}\left(\left|-\Omega(c+1)+\frac{1}{\Omega}\right|^2+\gamma^2\right)\left(\langle V_s^2\delta(\omega-\omega_s)\gamma_r^2+\langle V_{1n}^2\rangle\gamma^2\right)+ \\
&|\Omega|^2c^2\left(\langle V_s^2\delta(\omega-\omega_s)\gamma_r^2+\langle V_{2n}^2\rangle\gamma^2\right)+ \\
&\Omega^*c\gamma_r^2V_s^2\delta(\omega-\omega_s)\left((-\Omega(c+1)+\frac{1}{\Omega})\cos\phi+\gamma\sin\phi\right)+ \\
&\Omega c\gamma_r^2V_s^2\delta(\omega-\omega_s)\left((-\Omega^*(c+1)+\frac{1}{\Omega^*})\cos\phi+\gamma\sin\phi\right)
\end{aligned}$$

we have :

$$\langle V_{sn}^2\rangle=\frac{1}{2\pi}\int 4k_BT_srd\omega \tag{E.3}$$

$$\langle V_{1n}^2\rangle=\frac{1}{2\pi}\int 4k_BT_sR_1d\omega \tag{E.4}$$

$$\langle V_{2n}^2\rangle=\frac{1}{2\pi}\int 4k_BT_sR_2d\omega \tag{E.5}$$

So we have the first term:

$$\frac{4k_BT_s|r|\omega_0}{2\pi(1+2c)^2}\int\frac{\left(\left|-\Omega(c+1)+\frac{1}{\Omega}\right|^2+\gamma^2+|\Omega|^2c^2\right)\gamma_r^2}{|\tilde{\Omega}|^2}d\Omega \tag{E.6}$$

Second Equation:

$$\frac{V_s^2\gamma_r^2}{(1+2c)^2}\int\frac{\left(\left|-\Omega(c+1)+\frac{1}{\Omega}\right|^2+\gamma^2+|\Omega|^2c^2\right)\delta(\Omega-\Omega_s)}{|\tilde{\Omega}|^2}d\Omega \tag{E.7}$$

Third one:

$$\frac{4k_BT|R_1|\omega_0}{2\pi(1+2c)^2}\int\frac{\left|-\Omega(c+1)+\frac{1}{\Omega}\right|^2\gamma^2}{|\tilde{\Omega}|^2}d\Omega \tag{E.8}$$

Fourth one:

$$\frac{4k_BT|R_2|\omega_0}{2\pi(1+2c)^2}\int\frac{|\Omega|^2c^2\gamma^2}{|\tilde{\Omega}|^2}d\Omega \tag{E.9}$$

Fifth Equation:

$$\frac{c\gamma_r^2 V_s^2}{(1+2c)^2}\int d\Omega \frac{\delta(\Omega-\Omega_s)}{|\tilde{\Omega}|^2}\times \tag{E.10}$$
$$\left\{\Omega^*\left((-\Omega(c+1)+\frac{1}{\Omega})\cos\phi+\gamma\sin\phi\right)+\right.$$
$$\left.\Omega\left((-\Omega^*(c+1)+\frac{1}{\Omega^*})\cos\phi+\gamma\sin\phi\right)\right\}$$

Signal Parts:

$$\frac{V_s^2\gamma_r^2\left(\left|\Omega_s(c+1)+\frac{1}{\Omega_s}\right|^2+\gamma^2+|\Omega_s|^2c^2\right)}{\left(\prod_{j=1}^4(\Omega_s-\Omega_j-i\Delta)(\Omega_s^*-\Omega_j^*+i\Delta)\right)} \tag{E.11}$$

$$\frac{c\gamma_r^2V_s^2\left(\Omega_s^*(-\Omega_s(c+1)+\frac{1}{\Omega_s})\cos\phi+\gamma\sin\phi+c.c.\right)}{\left(\prod_{j=1}^4(\Omega_s-\Omega_j-i\Delta)(\Omega_s^*-\Omega_j^*+i\Delta)\right)} \tag{E.12}$$

Where c.c. is complex conjugate of the previous term. Noise part without signal noise, with just Gain and Loss noise sources.

$$\frac{4k_BT|R_1|\gamma^2\omega_0}{2\pi}\int\frac{\left|-\Omega(c+1)+\frac{1}{\Omega}\right|^2+\gamma^2}{\left(\prod_{j=1}^4(\Omega-\Omega_j-i\Delta)(\Omega^*-\Omega_j^*+i\Delta)\right)}\,d\Omega \tag{E.13}$$

$$\frac{4k_BT|R_2|\gamma^2c^2\omega_0}{2\pi}\int\frac{|\Omega|^2}{\left(\prod_{j=1}^4(\Omega-\Omega_j-i\Delta)(\Omega^*-\Omega_j^*+i\Delta)\right)}\,d\Omega \tag{E.14}$$

For computing signal to noise ratio (SNR) we just consider the noise in the gain side eq. (E.14) and the signal part eq. (E.11) and eq. (E.12). If $\omega_+\approx\omega_0$ then $\Omega_+\approx 1\approx\Omega_s$, by simplifying the noise and signal equations we have for noise RMS voltage:

$$\langle V_n^2\rangle=\frac{k_BT\gamma^2c^2}{2C\gamma_2}\frac{1}{(|\Omega_+|^2-|\Omega_-|^2)(|\Omega_+|^2+|\Omega_-|^2)\Delta} \tag{E.15}$$

And RMS signal is

$$\langle V_n^2\rangle=(V_s^2\gamma_r^2)\frac{\left(|\Gamma|^2+\gamma^2+|\Omega|_+^2c^2+c\left(\Omega^*(\Gamma\cos\phi+\gamma\sin\phi)+c.c.\right)\right)}{4|\Omega_+|^2((|\Omega_+|^2-|\Omega_-|^2)(|\Omega_+|^2+|\Omega_-|^2)\Delta^2} \tag{E.16}$$

So SNR is as follows

$$\langle V_n^2 \rangle = \frac{C\gamma_2 V_s^2 \gamma_r^2}{k_B T \gamma^2 c^2} \frac{1}{\Delta} \times \frac{\left(|\Gamma|^2 + \gamma^2 + |\Omega|_+^2 c^2 + c\left(\Omega^*(\Gamma\cos\phi + \gamma\sin\phi) + c.c.\right)\right)}{4|\Omega_+|^2} \tag{E.17}$$

Where $\Gamma = \Omega(1+c) - \frac{1}{\Omega}$.